# MASSACHUSETTS MATHEMATICS LEAGUE <br> CONTEST 5 - February 2023 <br> Practice Problems 

1A) Given: $f(x)=\frac{2}{9-x}$ Compute $\left(f^{-1}(-2)\right)^{3}$.
2A) Compute the sum of all two-digit positive integers for which twice the sum of the digits plus the product of the digits equals the integer.

3A) Angle $P$, angle $Q$, and a 240 degree angle are coterminal angles in standard position. Compute the ordered pair $(m \angle P, m \angle Q)$, if $2800^{\circ}<m \angle P<3160^{\circ}$ and $-3160^{\circ}<m \angle Q<-2800^{\circ}$.

4A) The flag shown at the right (drawn to scale) was immortalized by Francis_Scott Key during the bombardment of Fort McHenry on Sept 13, 1814. It is the only American flag that had more than 13 stripes, as Vermont and Kentucky had recently become the 14th and 15 th states, respectively.
Suppose the width (i.e., the shorter dimension) of this


American flag is 45 inches.
There are 8 short stripes and 7 long stripes.
The long stripes are 6 feet long.
All the stripes have the same width.
The blue star field ( 15 stars - one for each state) is 2 feet by 2 feet 9 inches.
What fraction of the total area of this flag is red?
5A) The ratio of the number of square units in the area of a circle to the number of units in its circumference is $5: 4$. Compute the length of the arc of a sector with a central angle of $22.5^{\circ}$.

6A) The sum of the first $k$ natural numbers is less than or equal to 1200 .
Compute the maximum possible value of $k$.

Team A)
The cubic equation $P(x)=2 x^{3}+9 x^{2}-5 x-36=0$ has roots $r, s$, and $t$. Each root of a new cubic equation $Q(x)=0$ is the sum of two roots of $P(x)=0$ minus the reciprocal of the third root.
Compute the sum of the roots of $Q(x)=0$.

## Answers:

1A) 1000
4A) $\frac{37}{90}$
2A) 127
5A) $\frac{5 \pi}{16}$
3A) $\left(3120^{\circ},-3000^{\circ}\right)$

Team A) $\quad-\frac{319}{36}$
The cubic equation $P(x)=2 x^{3}+9 x^{2}-5 x-36=0$ has roots $\boldsymbol{r}$, $\boldsymbol{s}$, and $\boldsymbol{t}$. Each root of a new cubic equation $Q(x)=0$ is the sum of two roots of $P(x)=0$ minus the reciprocal of the third root. Compute the sum of the roots of $Q(x)=0$.

Determining the actual roots of the cubic equation is not necessary. If you are familiar with Vieta's Formulas, which specify the relationship between the coefficients of a polynomial equation and its roots, skip the following derivation for the cubic equation:

$$
\begin{aligned}
& \text { If } \begin{aligned}
& P(x)=x^{2}+a x^{2}+b x+c=\left(x-r_{1}\right)\left(x-r_{2}\right)\left(x-r_{3}\right)=0 \\
&=\left(x^{2}-\left(r_{1}+r_{2}\right) x+r_{1} \cdot r_{2}\right)\left(x-r_{3}\right)=0 \\
&=x^{3}-\left(r_{1}+r_{2}\right) x^{2}+r_{1} \cdot r_{2} \cdot x-r_{3} \cdot x^{2}+\left(r_{1}+r_{2}\right) \cdot r_{3} \cdot x-r_{1} \cdot r_{2} \cdot r_{3}=0 \\
&=x^{3}-\left(r_{1}+r_{2}+r_{3}\right) x^{2}+\left(r_{1} r_{2}+r_{1} r_{3}+r_{2} r_{3}\right) x-\underline{r_{1} r_{2} r_{3}}=0, \\
& \text { then }\left\{\begin{array}{l}
a=-\left(r_{1}+r_{2}+r_{3}\right) \\
b=r_{1} r_{2}+r_{1} r_{3}+r_{2} r_{3} \\
c=-r_{1} r_{2} r_{3}
\end{array} .\right.
\end{aligned}
\end{aligned}
$$



## François Viète (1540-1603)

Dividing through by 2 so the lead coefficient of $P(x)=0$ is 1 , and applying the boxed relationships, we have $r+s+t=-\frac{9}{2}, r s+r t+s t=-\frac{5}{2}, r s t=18$.
The roots of $Q(x)=0$ are $\left(r+s-\frac{1}{t}\right),\left(r+t-\frac{1}{s}\right),\left(s+t-\frac{1}{r}\right)$.
Adding, the sum of these roots is $2(r+s+t)-\frac{r s+r t+s t}{r s t} \Rightarrow 2\left(-\frac{9}{2}\right)-\frac{-\frac{5}{2}}{18}=-9+\frac{5}{36}=-\frac{\mathbf{3 1 9}}{\mathbf{3 6}}$.

