MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - January 2023
Practice Problems

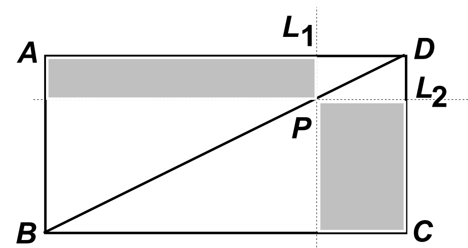
1A) The vertical line defined by $x+6=0$ intersects an ellipse defined by $\frac{(x+3)^2}{36}+\frac{(y-1)^2}{27}=1$ at points P and Q. Compute PQ.

2A) If phactor means to write as a sum of positive integers, the phactorization of an integer would not be unique. In how many different ways could 6 be phactored, if the order of the terms is irrelevant, i.e., $1+2+3$, $1+3+2$, and $3+2+1$ are not considered different?

3A) Solve for x over $0° \le x < 360°$: $\cos(2x) - 3\cos(x) = -2$

4A) There are four numbers less than 500 that are the product of exactly 4 distinct primes. Compute a positive integer solution of $x^2 - x = N$, where N is one of those four numbers.

5A) Perpendicular lines L_1 and L_2 intersect at interior point P on diagonal $\overline{BD}$ in rectangle $ABCD$. If $AB = 65$, $BC = 156$, and $PD = 52$, compute the sum of the areas of the shaded regions.



6A) Given: $x:y=3:8$, $y:z=6:19$
Compute z, if $x = 1.8$.

Team F)
The graphs of the following four equations intersect in exactly four points A, B, C, and D. A lies on 3 of the 4 graphs.

$$\begin{cases} y=-5.5x-12 \\ x-y+14=0 \\ 4x+7y=-1 \\ \frac{x}{2.25}+\frac{y}{1}=1 \end{cases}$$

Compute the value of $\min(AB, AC, AD)$.

Answers:

1A)	9	4A)	15 (for $N = 210$), or 22 (for $N = 462$)
2A)	10	5A)	4320
3A)	0°, 60°, 300°	6A)	15.2

Team F) $\frac{5}{7}\sqrt{97}$

The graphs of the following four equations intersect in exactly four points *A*, *B*, *C*, and *D*. *A* lies on 3 of the 4 graphs.

$$\begin{cases} y = -5.5x - 12 \\ x - y + 14 = 0 \\ 4x + 7y = -1 \\ \dfrac{x}{2.25} + \dfrac{y}{1} = 1 \end{cases}$$

Compute the value of $\min(AB, AC, AD)$**.**

Four points of intersection imply two possibilities:

- two pairs of parallel lines
- three lines passing through a common point and a fourth line that intersects the first three lines

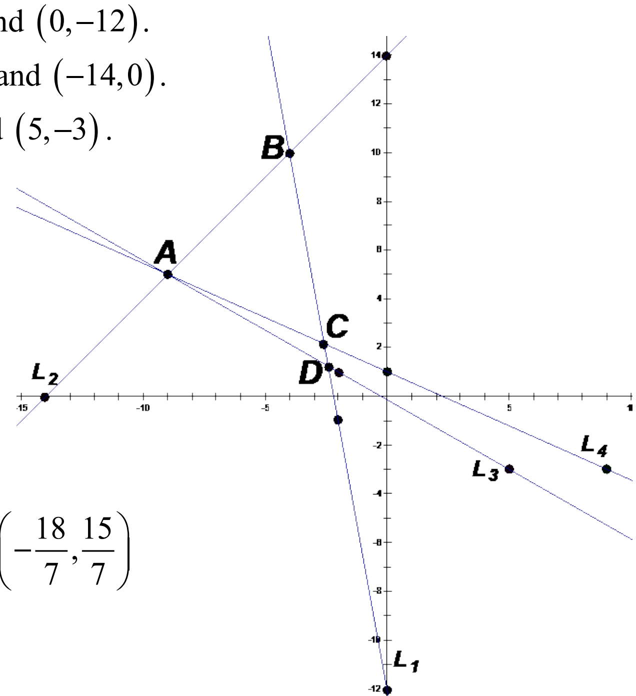
Since, by inspection, the slopes the first three lines are $-5.5, 1$, and $-\frac{4}{7}$, we are dealing with the second case. A sketch will give us a clue about which calculations are absolutely necessary.

The graph of the first equation is a line through $(-2,-1)$ and $(0,-12)$.

The graph of the second equation is a line through $(0,14)$ and $(-14,0)$.

The graph of the third equation is a line through $(-2,1)$ and $(5,-3)$.

Since the fourth equation is in intercept-intercept form, we know it passes through $(2.25,0)$ and $(0,1)$, or, using the slope $m = -\frac{4}{9}$ through $(9,-3)$. Since $\angle ACD$ is obtuse, $AD > AC$, $\min(AB, AC, AD)$ is either AB or AC.



$$\begin{cases} x - y + 14 = 0 \\ 4x + 7y = -1 \end{cases} \Rightarrow A(-9,5)$$

$$\begin{cases} y = -5.5x - 12 \\ x - y + 14 = 0 \end{cases} \Rightarrow B(-4,10) \qquad \begin{cases} y = -5.5x - 12 \\ \dfrac{x}{2.25} + \dfrac{y}{1} = 1 \end{cases} \Rightarrow C\left(-\frac{18}{7}, \frac{15}{7}\right)$$

$$AB^2 = (-4+9)^2 + (10-5)^2 = 50$$

$$AC^2 = \left(-9 + \frac{18}{7}\right)^2 + \left(5 - \frac{15}{7}\right)^2 = \frac{45^2 + 20^2}{7^2} = \frac{2425}{49} \approx 49.5$$

Thus, $\min(AB, AC, AD) = AC = \frac{1}{7}\sqrt{2425} = \frac{1}{7}\sqrt{25 \cdot 97} = \underline{\boldsymbol{\frac{5}{7}\sqrt{97}}}$.

Original 4A) incorrectly stated that there are 10 numbers less than 1200 that....
There are actually 20 such numbers

$2 \cdot 3 \cdot 5 \cdot 7 = 210 \Rightarrow 15 \cdot -14$

$2 \cdot 3 \cdot 5 \cdot 11 = 330$

$2 \cdot 3 \cdot 5 \cdot 13 = 390$

$2 \cdot 3 \cdot 5 \cdot 17 = 510$

$2 \cdot 3 \cdot 5 \cdot 19 = 570$

$2 \cdot 3 \cdot 5 \cdot 23 = 690$

$2 \cdot 3 \cdot 5 \cdot 29 = 870 \Rightarrow 30 \cdot -29$

$2 \cdot 3 \cdot 5 \cdot 31 = 930 \Rightarrow 31 \cdot -30$

$2 \cdot 3 \cdot 5 \cdot 37 = 1110$

$\cancel{2 \cdot 3 \cdot 5 \cdot 41 = 1230}$

$2 \cdot 3 \cdot 7 \cdot 11 = 462 \Rightarrow 22 \cdot -21$

$2 \cdot 3 \cdot 7 \cdot 13 = 546$

$2 \cdot 3 \cdot 7 \cdot 17 = 714$

$2 \cdot 3 \cdot 7 \cdot 19 = 798$

$2 \cdot 3 \cdot 7 \cdot 23 = 966$

$\cancel{2 \cdot 3 \cdot 7 \cdot 29 = 1218}$

$2 \cdot 3 \cdot 11 \cdot 13 = 858$

$2 \cdot 3 \cdot 11 \cdot 17 = 1122 \Rightarrow 34 \cdot -33$

$\cancel{2 \cdot 3 \cdot 11 \cdot 19 = 1254}$

$\cancel{2 \cdot 3 \cdot 13 \cdot 17 = 1326}$

$2 \cdot 5 \cdot 7 \cdot 11 = 770$

$2 \cdot 5 \cdot 7 \cdot 13 = 910$

$2 \cdot 5 \cdot 7 \cdot 17 = 1190 \Rightarrow 35 \cdot -34$

$3 \cdot 5 \cdot 7 \cdot 11 = 1155$