MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - December 2022
Practice Problems

1A) The hypotenuse $\overline{AB}$ of right triangle ABC has length 12, and the legs have lengths in a ratio of 7: 24. If angle A is the larger acute angle in ΔABC, compute $\sin^2 A + \cos^2 B$.

2A) My boss Phineaus T. Bluster reduced my salary by 14 2/7 %.
If, after January 1st, he were to increase my reduced salary by k%, my salary would be returned to its original amount. Compute the exact value of k.

3A) Given: $A(4, -56)$ and $B(100, -16)$
Determine integers x and y for which $P(x, y)$ is closest to the origin of all lattice points on $\overleftrightarrow{AB}$ in quadrant 1.

4A) Given: $f(x) = 4^x$
Compute $\dfrac{f(x+1) - f(x)}{f(x-1) - f(x)}$.

5A) A buoy has $\frac{1}{4}$ of its length in the water, $\frac{1}{6}$ of its length in the sand, and 21 feet in the air. Compute the total length of the buoy (in feet).

6A) If a polygon had 5 more sides, it would have 75 more diagonals.
How many sides does this polygon have?

Team A)
A triangle has sides of length 2, 5 and 6. A cevian drawn to the longest side divides that side into segments in a 2 : 1 ratio. Compute all possible lengths of this cevian. [Note: a cevian is a line segment from a vertex of a triangle to a point on the opposite side.]

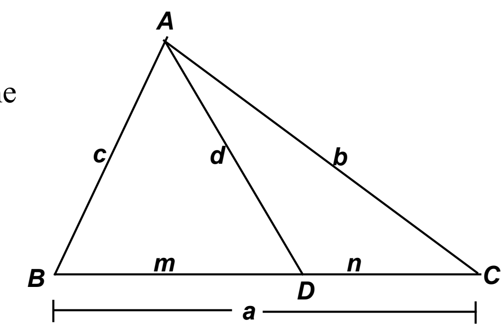
Answers:

1A)	$\frac{1152}{625}$	4A)	-4
2A)	$16\frac{2}{3}$	5A)	36
3A)	$x = 148, y = 4$	6A)	14

Team A) $\sqrt{3}, \sqrt{10}$

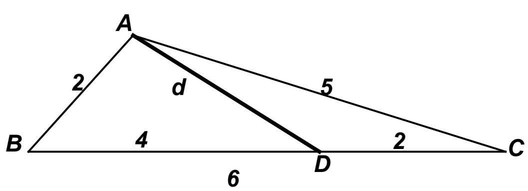
A triangle has sides of length 2, 5 and 6. A cevian drawn to the longest side divides that side into segments in a 2 : 1 ratio. Compute all possible lengths of this cevian. [Note: a cevian is a line segment from a vertex of a triangle to a point on the opposite side.]

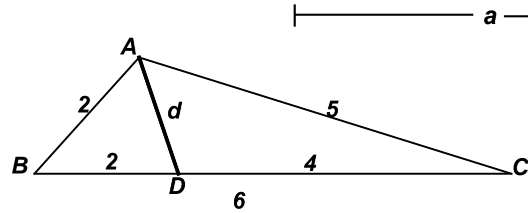
The segments on the longest side have lengths 2 and 4.
Using Stewart's theorem to solve this problem is a breeze.
Here's the theorem: $man + d^2a = b^2m + c^2n$



As long as the triangle is labeled as in the diagram at the right, the following mnemonic will help you remember the equation.
A **man** and his **dad** put a **b**o**mb** in the **cnc** (pronounced 'sink').
man + dad = bmb + cnc $\Leftrightarrow$ $man + d^2a = b^2m + c^2n$

The two possibilities are:





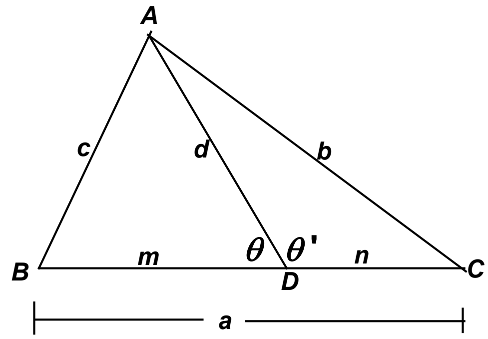
$$4 \cdot 6 \cdot 2 + 6d^2 = 5^2 \cdot 4 + 2^2 \cdot 2$$
$$6d^2 = 100 + 8 - 48 = 60$$
$$d^2 = 10 \Rightarrow d = \underline{\sqrt{\mathbf{10}}}$$

$$2 \cdot 6 \cdot 4 + 6d^2 = 5^2 \cdot 2 + 2^2 \cdot 4$$
$$6d^2 = 50 + 16 - 48 = 18$$
$$d^2 = 3 \Rightarrow d = \underline{\sqrt{\mathbf{3}}}$$

The proof of Stewart's Theorem is given on the next page.
Hint: Apply the Law of Cosines to ΔBAD and to ΔCAD, using the supplementary angles at D.
Try proving it yourself or discussing a proof with your teammates and/or coach before peeking.

Symbolically, we wish to show that

$\boxed{man+d^2a=b^2m+c^2n}$.



In actual words, it sounds like an episode of the "Twilight Zone".
The three-way product of the length of the divided side and the lengths of the two segments on that side PLUS the product of the square of the length of the cevian and the total length of the divided side equals the product of the square of the length of one of the undivided sides of the triangle and the length of the remote segment on the divided side PLUS the product of the square of the other undivided side and the length of its corresponding remote segment.

Makes you appreciate the mnemonic: A **man** and his **dad** put a **b**o**mb** in the **cnc** ("sink").

Here's the proof: (It's surprisingly short.)
Let θ and θ' denote the supplementary angles at *D*, as indicated in the diagram.
Using the Law of Cosines,
In $\Delta BAD,\ c^2 = m^2 + d^2 - 2md\cos\theta$
In $\Delta CAD,\ b^2 = n^2 + d^2 - 2nd\cos\theta' = n^2 + d^2 - 2nd\cos(180° - \theta) = n^2 + d^2 + 2nd\cos\theta$
Note that we have applied the reduction formula $\cos(180° - \theta) = -\cos\theta$, which loosely translates to the cosines of supplementary angles are equal in absolute value, but opposite in sign.
If θ is acute, then $180° - \theta$ is obtuse, and we would expect the cosine to be negative.

The plan is to multiply the first equation by *n*, the second equation by *m*, and add.

$$\begin{cases} c^2n = m^2n + d^2n - 2mnd\cos\theta \\ b^2m = n^2m + d^2m + 2mnd\cos\theta \end{cases}$$

$$\Rightarrow c^2n + b^2m = (m^2n + n^2m) + (d^2n + d^2m) = mn(m+n) + d^2(m+n) = mna + d^2a$$

$$\Rightarrow \boxed{man+d^2a=b^2m+c^2n}.$$

There we have it!
Q.E.D.