MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - February 2022
Practice Problems

1A) A linear function $y = mx + b$ is defined by the parametric equations $\begin{cases} x = \dfrac{2-3t}{4} \\ y = \dfrac{4-t}{5} \end{cases}$.

For $t = a$, the linear function passes through $(5,2)$; for $t = b$, the linear function passes through $(-10,-2)$. Compute $b - a$.

2A) The sum of the digits of a two-digit number is 11.
Compute the sum of all such numbers that are prime.

3A) Compute: $\operatorname{Sin}^{-1}\left(\cos\left(\dfrac{5\pi}{3}\right)\right) - \operatorname{Cos}^{-1}\left(\sin\left(\dfrac{7\pi}{6}\right)\right)$.

4A) For Halloween, I bought a package of 1128 jelly beans (JBs), some red and some green. If the package had contained 12 more red JBs and 12 fewer green JBs, the ratio of red JBs to green JBs would have been 5 : 7. How many red JBs from the original mixture did I eat, if the red-to-green ratio dropped to 3 : 5?

5A) A triangle with angles whose measures are in a 5 : 8 : 8 ratio is inscribed in a circle. To the nearest integer, compute the number of degrees in the measure of the arc subtended by the vertex angle of this triangle.

6A) Given: $(6,8)$, $(8,15)$, $(10,24)$, $(12,35)$, $(14,48)$, $(16,63)$, ..., (A,B)
The following statements are true about this sequence:
Each ordered pair represents the lengths of the legs of a right triangle with integer perimeter.
The sequence of first coordinates is an increasing sequence,
as is the sequence of second coordinates.
The absolute value of the difference between consecutive first coordinates is 2.
The absolute value of the difference between consecutive second coordinates is increasing by a constant.
Compute the minimum value of $A + B$, given that the area of the right triangle with legs of length A and B is greater than 2022.

Team F)
For positive integers h and k, $(t_1, t_2, t_3) = (3 + 2k, 4 + h, 1 + k)$ are the first three terms of a geometric sequence. Compute the minimum value of $h + k$.

Answers:

1A)	20	4A)	56
2A)	159	5A)	86
3A)	$-\frac{\pi}{2}$	6A)	194
Team F)	5		

For <u>positive</u> integers *h* and *k*, $(t_1,t_2,t_3)=(3+2k,4+h,1+k)$ are the first three terms of a geometric sequence. Compute the <u>minimum</u> value of $h+k$.

Given: $(t_1,t_2,t_3)=(3+2k,4+h,1+k)$

In a geometric progression, there is a constant ratio between successive terms $\Rightarrow$

$\frac{t_2}{t_1}=\frac{t_3}{t_2}\Rightarrow t_1t_3=t_2^{\,2}\Rightarrow(3+2k)(1+k)=(4+h)^2$

$\Rightarrow 2k^2+5k+\left(3-(4+h)^2\right)=0$

Treating *h* as a constant, this is a quadratic equation in *k*.

Applying the quadratic formula to this second-degree trinomial, we have

$$k=\frac{-5+\sqrt{25-8\left(3-(h+4)^2\right)}}{4}=\boxed{\frac{-5+\sqrt{1+8(h+4)^2}}{4}}$$

For *k* to be an integer, the radicand must be a perfect square.

$h=1,\ 2,\ \ldots\Rightarrow 1+8(h+4)^2=201,\ 289,\ldots$

Since $289=17^2$, $k=\frac{-5+17}{4}=3\Rightarrow h+k=\underline{\mathbf{5}}$.

Check: $(t_1,t_2,t_3)=(9,6,4)\Rightarrow$ common multiplier of $\frac{2}{3}$.

<u>FYI</u>:

Fortunately, a small *h*-value gave rise to a perfect square radicand, and the boxed quotient produced an integer value.

With the help of a calculator, one can determine that the next *h*-value that produces a perfect square is $h=31$ [9801= 99^2].

However, $k=\frac{-5+99}{4}$ fails. After another long sequence of failing *h*-values,

$h=200\Rightarrow 1+8(h+4)^2=332929=577^2$ and $\frac{-5+577}{4}=143$.

$(h,k)=(200,143)\Rightarrow(t_1,t_2,t_3)=(289,204,144)$ and $289\cdot144=17^2 12^2=(17\cdot12)^2=204^2$

The common multiplier is $\frac{12}{17}$.