MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - December 2021
Practice Problems

1A) Forrest Gump walked 1 mile east and pounded a stake in the ground.
He then walked 3 miles north, 6 miles northeast, 6 miles southeast, and finally 6 miles south.
How many miles is his final position from the stake?

2A) $\dfrac{n!}{3^{13}}$ is an integer. Compute the minimum value of the integer *n*.
Note: $n! = n(n-1)(n-2)\cdot \ldots \cdot 2 \cdot 1$

3A) Assume that *O* is the center of the coordinate system.
A circle has center *P* in quadrant IV at (h,k), where *h* and *k* are integers and $|h|<|k|$.
If the circumference of circle *P* is 5π, and $OP = 13$, determine the equation of circle *P* in $(x-h)^2 + (y-k)^2 = r^2$ form.

4A) Determine all values of *x* for which $(625)^x - 150(25)^x + 3125 = 0$.

5A) Given: $a:b = 3:8$, $b:c = 5:k$
Compute the minimum positive integer value of *k* for which $\dfrac{a+c}{b+c} > \dfrac{4}{5}$.

6A) *ABCD* is a square with integer side-lengths. Its area, a multiple of 7, is greater than 500. No smaller square satisfies these requirements. *PQRS* is a non-square rectangle with integer side-lengths whose area is the same as the area of *ABCD*, and whose perimeter *p* is closest to the perimeter of *ABCD*. Compute *p*.

Team E)
A is the set of all positive proper reduced fractions with denominators of at most 6.
S is the sequence formed from the elements in set *A*, arranged in increasing order.
Let *k* be the number of fractions in *S*. Let *j* be the sum of 4 particular terms in *S*, namely, $t_1 + t_4 + t_7 + t_{10}$.
Compute the ordered pair (k, j).

Answers:

1A) 9

2A) 27

3A) $(x-5)^2+(y+12)^2=\dfrac{25}{4}$

4A) 1, 1.5 (or equivalent)

5A) 11

6A) 130

Team E) $\left(11, \dfrac{19}{10}\right)$ or $(11, 1.9)$

It's easy to determine the number of reduced fractions.
With denominators of 2, ..., 6, we have 1, 2, 2, 4, and 2 reduced fractions, respectively, for a total of 11. Putting them in sequence is a little trickier.

Start with boundaries of $\frac{0}{1}$ and $\frac{1}{1}$.

Adding numerators and adding denominators, we have our first fraction: $\frac{1}{2}$.

Insert this fraction and continue adding numerators and adding denominators.

$\frac{0}{1}, \mathbf{\frac{1}{3}}, \frac{1}{2}, \mathbf{\frac{2}{3}}, \frac{1}{1}$ (Note the sequence of 3 fractions is in increasing order.)

Continuing to add,

$\frac{0}{1}, \mathbf{\frac{1}{4}}, \frac{1}{3}, \mathbf{\frac{2}{5}}, \frac{1}{2}, \mathbf{\frac{3}{5}}, \frac{2}{3}, \mathbf{\frac{3}{4}}, \frac{1}{1}$ (still in increasing order!)

At this point, the sequence has grown to 7 terms, with 4 terms still to be inserted.
We could mindlessly continue creating additional fractions by adding numerators and adding denominators, but that would be tedious and we would be inserting fractions with denominators greater than 6 (which would have to be eliminated).

It's easier to note that the fractions still to be inserted are $\left\{\frac{1}{5}, \frac{1}{6}, \frac{4}{5}, \frac{5}{6}\right\}$.

The first two fractions must be inserted before $\frac{1}{4}$, and the last two after $\frac{3}{4}$.

Thus, the complete sequence is

$\frac{0}{1}, \boxed{\frac{1}{6}}, \frac{1}{5}, \frac{1}{4}, \boxed{\frac{1}{3}}, \frac{2}{5}, \frac{1}{2}, \boxed{\frac{3}{5}}, \frac{2}{3}, \frac{3}{4}, \boxed{\frac{4}{5}}, \frac{5}{6}, \frac{1}{1}$.

The terms whose sum we seek have been boxed: $\frac{1}{6}+\frac{1}{3}+\left(\frac{3}{5}+\frac{4}{5}\right)=\frac{5+10+42}{30}=\frac{57}{30}=\frac{19}{10}$.

Thus, $(k, j)=\underline{\left(\mathbf{11, \frac{19}{10}}\right)}$ or $\underline{(\mathbf{11, 1.9})}$.