MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - November 2021
Practice Problems

1A) Given: $z = 2 - i$
Five times the sum of the reciprocal of z and the congugate of z equals $a + bi$.
Compute the ordered pair (a, b).

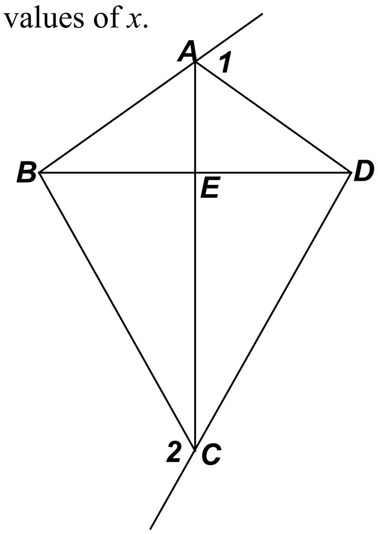
2A) Compute the <u>sum</u> of all integer values of x for which $x^2 - 4x \le 2021$.

3A) The diagonals of a rhombus have lengths of 5 and 10.
Compute P, the perimeter of this rhombus.

4A) Compute the <u>largest</u> 2-digit integer k for which $3k$ has exactly 12 factors.

5A) $A = x^2 - x$ is the largest number of degrees in the measure of angle A in the interval $0° < A \le 360°$ for which $2\cos A + 1 = 0$. Compute <u>all</u> possible values of x.

6A) $ABCD$ is a kite, its axis of symmetry is $\overline{AC}$,
$m\angle 1 = 84°$, and $m\angle 2 = 110°$.
$\angle 1$ forms a linear pair with $\angle BAD$.
$\angle 2$ forms a linear pair with $\angle BCD$.
Compute the number of degrees in $m\angle BDA + m\angle DBC$.



Team B)
The sum of a two-digit natural number and its reversal is 66.
The sum of all such numbers is T. How many natural numbers less than T are relatively prime to T, i.e., have no factors in common with T (other than 1)?

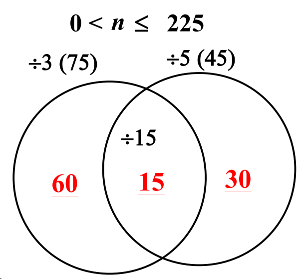
Note: There is no requirement that the reversal will also be a two-digit number.

Answers:

1A)	(12, 6)	4A)	98
2A)	182	5A)	-15, 16
3A)	$10\sqrt{5}$	6A)	97

Team B) 120

The sum of a two-digit natural number and its reversal is 66.
The sum of all such numbers is *T*. How many natural numbers less than *T* are relatively prime to *T*, i.e., have no factors in common with *T* (other than 1)?



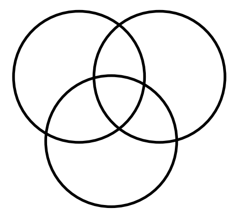
$(10x+y)+(10y+x)=66 \Rightarrow x+y=6 \Rightarrow 60,\ 51,\ 15,\ 42,\ 24,\ 33$

$\Rightarrow\ T=225=3^2\cdot 5^2$

A Venn diagram helps avoid over-counting multiples of 3 and 5.

$225-(60+15+30)=\underline{\mathbf{120}}$

If the prime factorization had 3 distinct factors, to separate the overlaps, the Venn Diagram would look like this:



For 4 or more factors, drawing a Venn Diagram is more challenging or impossible.
A different approach would be helpful.

Alternately, the φ function (pronounced 'fee') counts all the natural numbers less than *n* that are relatively prime to *n*. This function is fast and easy to use if the prime factorization of *n* is known.

If $n=p_1^{e_1}\cdot p_2^{e_2}\cdot\ldots\cdot p_k^{e_k}$, then $\varphi(n)=n(1-p_1)(1-p_2)\cdot\ldots\cdot(1-p_k)$

For $225=3^2 5^2$, we have $225\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)=225\cdot\frac{2\cdot 4}{3\cdot 5}=15\cdot 8=\underline{\mathbf{120}}$.

It's not hard to see why this function does the job. It says:
One third of the numbers are divisible by 3; eliminate them: 225 - 75 = 150.
One fifth of the remaining numbers are divisible by 5; eliminate them: 150 - 30 = **<u>120</u>**.
What's left will have no factors in common with 225.