MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2020
ROUND 1 ALGEBRA 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) (_____________ , _____________)

B) (_____________ , _____________)

C) ______________________________

A) Compute the ordered pair (x, y) of positive integers for which $\begin{cases} x^2 + 2xy + 3y^2 = 57 \\ 2x^2 + 4xy + 5y^2 = 110 \end{cases}$.

B) If $\dfrac{\det\begin{bmatrix} c_1 & 2 \\ c_2 & -8 \end{bmatrix}}{\det\begin{bmatrix} 5 & 2 \\ 1 & -8 \end{bmatrix}} = 3$ and $c_1 + c_2 = 33$, compute the ordered pair (c_1, c_2).

C) For some positive integers k and N, the line $L_1 = \left\{(x, y) \mid \dfrac{x}{2k-1} - \dfrac{y}{k-2} = N\right\}$ is $3\sqrt{10}$ units from the line $L_2 = \left\{(x, y) \mid \dfrac{x}{4(k-8)} + \dfrac{y}{4(6-k)} = c\right\}$. If L_1 passes though the point $P(9, -3)$, compute all possible ordered triples (k, N, c).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2020 SOLUTION KEY

Round 1

A) Doubling both sides of the first equation and subtracting, we have
$y^2 = 114 - 110 \Rightarrow y = 2$ (since $x, y > 0$).
Substituting in the first equation,
$x^2 + 4x + 12 = 57 \Leftrightarrow x^2 + 4x - 45 = (x+9)(x-5) = 0 \Rightarrow x = 5$.
Thus, $(x, y) = \underline{\mathbf{(5,2)}}$.

B) $$\frac{\det\begin{bmatrix} c_1 & 2 \\ c_2 & -8 \end{bmatrix}}{\det\begin{bmatrix} 5 & 2 \\ 1 & -8 \end{bmatrix}} = 3 \Leftrightarrow \frac{-8c_1 - 2c_2}{-42} = 3 \Leftrightarrow 4c_1 + c_2 = 63.$$
Substituting for c_2, $4c_1 + (33 - c_1) = 63 \Rightarrow 3c_1 = 30 \Leftrightarrow c_1 = 10$.
Thus, $(c_1, c_2) = \underline{\mathbf{(10,\ 23)}}$.

C) Since lines L_1 and L_2 must be parallel, their slopes must be equal.
The slope of L_1 is $\dfrac{k-2}{2k-1}$ and the slope of L_2 is $-\dfrac{6-k}{k-8} = \dfrac{k-6}{k-8}$.
Equating, $\dfrac{k-2}{2k-1} = \dfrac{k-6}{k-8}$
$\Rightarrow k^2 - 10k + 16 = 2k^2 - 13k + 6 \Rightarrow k^2 - 3k - 10 = (k-5)(k+2) = 0 \Rightarrow k = 5, \cancel{-2}$.

$k = 5 \Rightarrow L_1 : \dfrac{x}{9} - \dfrac{y}{3} = N \Leftrightarrow x - 3y = 9N$ and $k = 5 \Rightarrow L_2 : \dfrac{x}{-3} + \dfrac{y}{1} = 4c \Leftrightarrow x - 3y + 12c = 0$.
Plugging in the coordinates of point P, $9N = 9 - 3(-3) \Rightarrow N = 2$.
The distance between the parallel lines is $\dfrac{|1\cdot(9) - 3\cdot(-3) + 12c|}{\sqrt{1^2 + 3^2}} = 3\sqrt{10} \Rightarrow |18 + 12c| = 30$.
$3 + 2c = \pm 5 \Rightarrow c = \dfrac{-3 \pm 5}{2} = 1, -4$.
Thus, $(k, N, c) = \underline{\mathbf{(5,2,1), (5,2,-4)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2020
ROUND 2 ALGEBRA 1: EXPONENTS AND RADICALS

ANSWERS

A) (____________ , ____________)

B) ______________________________

C) ______________________________

A) For a <u>minimum</u> positive integer value of *x*, $\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{x}{3}\right)^2}$ represents a <u>rational</u> number *k*.

Compute the ordered pair (x,k).

B) Compute the <u>largest</u> integer *x* for which $2^{x+2}-2^{x+6}-2^x+2^{x+7}+2^{x+3}<2020$.

C) Given: $y=\sqrt{3+2x-x^2}$

Let $A=x_{\max}-x_{\min}$, $B=-A$.

Compute $\left(y_{\min}\right)^A+\left(y_{\max}\right)^B$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2020 SOLUTION KEY

Round 2

A) $\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{x}{3}\right)^2}=\sqrt{\frac{9}{4}+\frac{x^2}{9}}=\sqrt{\frac{81+4x^2}{36}}=\frac{\sqrt{81+4x^2}}{6}$.

Testing, $x=1,2,3,\ldots \Rightarrow 81+4x^2=$ ~~85~~, ~~97~~, ~~117~~, ~~145~~, ~~181~~, $225=15^2$.

Thus, for $x=6$, $k=\frac{\sqrt{225}}{6}=\frac{15}{6}=\frac{5}{2}\Rightarrow \underline{\left(\mathbf{6,\frac{5}{2}}\right)}$.

B) $2^{x+2}-2^{x+6}-2^x+2^{x+7}+2^{x+3}=2^x\left(2^2-2^6-1+2^7+2^3\right)=2^x(4-64-1+128+8)=75\left(2^x\right)$.

$2^x<\frac{2020}{75}=\frac{404}{15}=27^-\Rightarrow x_{\max}=\underline{\mathbf{4}}$.

C) By definition, $y=\sqrt{3+2x-x^2}\geq 0$ and defined only if the radicand is non-negative.

$3+2x-x^2\geq 0\Leftrightarrow x^2-2x-3=(x-3)(x+1)\leq 0\Leftrightarrow -1\leq x\leq 3$.

$A=x_{\max}-x_{\min}=3-(-1)=4\Rightarrow B=-4$

Clearly, $y_{\min}=0$. By symmetry, $y_{\max}$ occurs at $x=\frac{-1+3}{2}=1$. Thus, $y_{\max}=\sqrt{3+2-1}=+2$,

resulting in $(y_{\min})^A+(y_{\max})^B=0^4+2^{-4}=\underline{\mathbf{\frac{1}{16}}}$.

Alternately, squaring both sides,

$y=\sqrt{3+2x-x^2}\Rightarrow y^2=3+2x-x^2\Rightarrow\left(x^2-2x+1\right)+y^2=3+1\Rightarrow(x-1)^2+y^2=4$.

Thus, the original equation represents a semi-circle above the *x*-axis.

Since the center is at (1,0) and radius has length 2, we have

$x_{\max}=3,\ x_{\min}=-1,\ y_{\min}=0, y_{\max}=2, A=3-(-1)=4,\ B=-4$.

Therefore, $(y_{\min})^A+(y_{\max})^B=(0)^4+(2)^{-4}=\underline{\mathbf{\frac{1}{16}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2020
ROUND 3 TRIGONOMETRY: ANYTHING

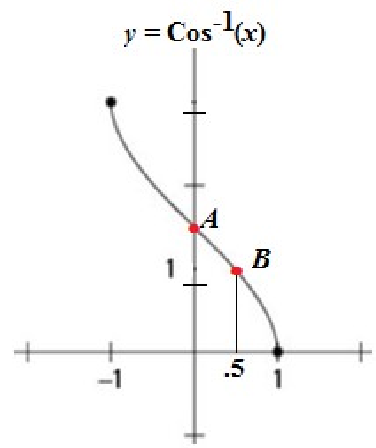
ANSWERS

A) (_____________ , _____________)

B) (_____________ , _____________)

C) ______________________________

A) Compute (p,q), the rectangular coordinates of the midpoint of $\overline{AB}$.



B) A small square plot of land is surrounded by a picket fence. There are 4 equally spaced pickets on each side, that is, 4 pickets and 3 gaps on each side. The width of each picket is given by $\cos x°$ and the space between the pickets is given by $\sin x°$ for some small value of *x*. The perimeter of the square can be written as $P\sin(Q+x)°$, where $Q = Sin^{-1}(R)$.

Compute the ordered pair (P,R).

C) Given: $\sin(2x) = \sin x - \cos x$

Compute *h*, the length of the hypotenuse of a right triangle whose legs have lengths $\sin(2x)$ and $\csc(2x)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2020 SOLUTION KEY

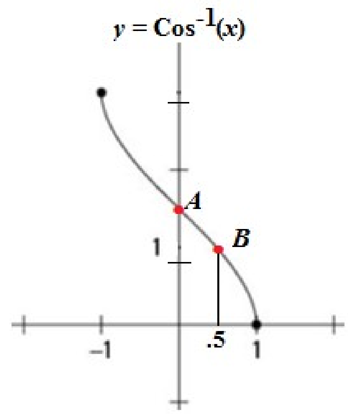
Round 3

A) Clearly, the graph indicates that q <u>must</u> be expressed in radians.

$\cos\left(\frac{\pi}{3}\right)=\frac{1}{2} \Rightarrow \text{Cos}^{-1}(.5)=\frac{\pi}{3}$, $\cos\left(\frac{\pi}{2}\right)=0 \Rightarrow \text{Cos}^{-1}(0)=\frac{\pi}{2}$.

The midpoint of a segment with endpoints at $\left(\frac{1}{2},\frac{\pi}{3}\right)$ and $\left(0,\frac{\pi}{2}\right)$

is $\left(\frac{\frac{1}{2}+0}{2},\frac{\frac{\pi}{3}+\frac{\pi}{2}}{2}\right)=\underline{\left(\mathbf{\frac{1}{4},\frac{5\pi}{12}}\right)}$.



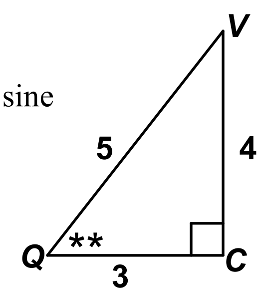
B) Since each side of the square is $4\cos x+3\sin x$, the perimeter is $4(4\cos x+3\sin x)$.

The parenthesized sum looks like the expansion of $\sin(\alpha+\beta)$, but sine and cosine values cannot be larger than 1. Rewriting the perimeter expression, we have

$4(4\cos x+3\sin x)=20\left(\frac{4}{5}\cos x+\frac{3}{5}\sin x\right)$, and the fractional values are the sine and the cosine values for the angle shown at the right.

Now, we have $20(\sin Q\cos x+\cos Q\sin x)=20\sin(Q+x)$ and

$(P,R)=\underline{\left(\mathbf{20,\frac{4}{5}}\right)}$.



C) Squaring both sides, we have

$\sin^2(2x)=\sin^2 x-2\sin x\cos x+\cos^2 x=1-2\sin x\cos x=1-\sin(2x)$.

Transposing terms ($\sin^2(2x)+\sin(2x)-1=0$), and using the quadratic formula,

$\sin(2x)=\frac{-1\pm\sqrt{1+4}}{2}=\frac{\sqrt{5}-1}{2}$, ~~$\frac{-\sqrt{5}-1}{2}$~~ $\Rightarrow \csc(2x)=\frac{2}{\sqrt{5}-1}=\frac{\sqrt{5}+1}{2}$.

Now $h=\sqrt{\left(\frac{\sqrt{5}+1}{2}\right)^2+\left(\frac{\sqrt{5}-1}{2}\right)^2}=\sqrt{\frac{6+2\sqrt{5}}{4}+\frac{6-2\sqrt{5}}{4}} \Rightarrow h=\underline{\sqrt{\mathbf{3}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 4 ALGEBRA 1: ANYTHING

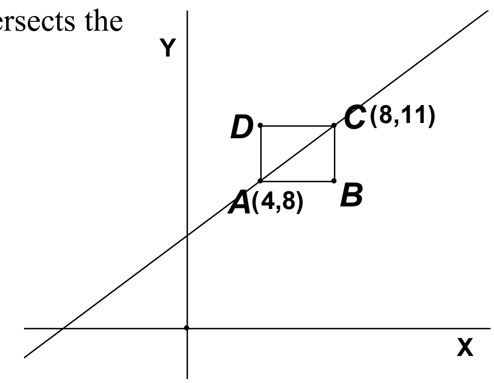
ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) The line containing diagonal $\overline{AC}$ of rectangle $ABCD$ intersects the axes at $(a,0)$ and $(0,b)$
Compute $a+b$.



B) Six less than twice a two-digit natural number equals 6 more than half the reversal of the two-digit natural number. Compute <u>all</u> two-digit natural numbers with this property.
If no numbers have this property, specify <u>none</u>.

C) For some positive integers A, the quadratic equation $(2x+1)(x-1) = A$ has rational roots x_1 and x_2, and the value of the discriminant is greater than 2020. Compute the minimum value of $|x_1 - x_2|$.

MASSACHUSETTS MATHEMATICS LEAGUE
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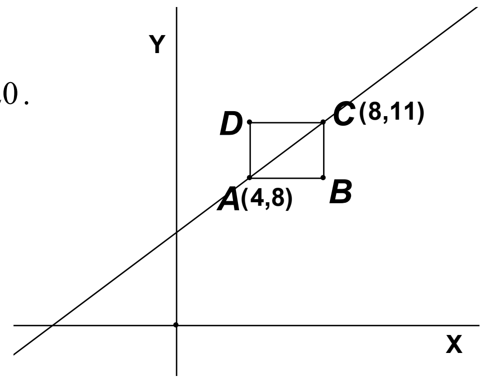
Round 4

A) The equation of $\overleftrightarrow{AC}$ is

$(y-8)=\frac{11-8}{8-4}(x-4) \Leftrightarrow 4y-32=3x-12 \Leftrightarrow -3x+4y=20$.



The x-intercept is $\frac{20}{-3}$. The y-intercept is $\frac{20}{4}$.

Thus, $a+b=-\frac{20}{3}+5=\underline{-\frac{\mathbf{5}}{\mathbf{3}}}$.

B) Let the original number be $10x+y$. Then:

$2(10x+y)-6=\frac{10y+x}{2}+6$

$\Leftrightarrow 4(10x+y)-12=10y+x+12 \Leftrightarrow \boxed{39x=6y+24=6(y+4)}$.

Regardless of the value of y, the RHS will be even, so the minimum x-value is 2.

$78=6(y+4) \Leftrightarrow y+4=13 \Rightarrow y=9$ and the original number can only be **<u>29</u>**.

Check: $2(29)-6=\frac{92}{2}+6=52$.

Alternately, divide both sides of the boxed equation by 3: $13x=2(y+4)$

Since x must be even, we have $x=2 \Rightarrow y=9$. Since $y \le 9$, there are no additional solutions.

C) $(2x+1)(x-1)=A \Leftrightarrow 2x^2-x-(1+A)=0$. The discriminant is $1+8(1+A)=8A+9$.

Rational roots require that the value of this expression be a perfect square.

$A=2,5,9,14,\ldots \Rightarrow 8A+9=5^2, 7^2, 9^2, 11^2, \ldots$. Thus, the discriminant gives us a sequence of the squares of consecutive odd integers. The minimum perfect square greater than 2020 is $45^2 = 2025$.

Applying the Quadratic Formula, this gives roots $\frac{1\pm 45}{4}=\frac{23}{2}$, -11, and

$|x_1-x_2|=\left|\frac{23}{2}-(-11)\right|$ or $\left|-11-\frac{23}{2}\right| \Rightarrow \underline{\frac{\mathbf{45}}{\mathbf{2}}}$ or **<u>22.5</u>**.

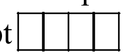
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2020
ROUND 5 PLANE GEOMETRY: ANYTHING

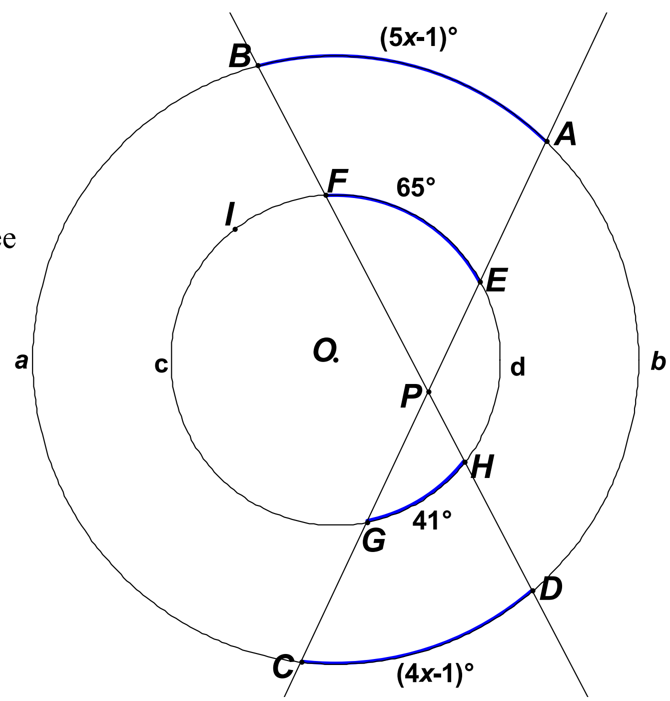
ANSWERS

A) (_____________ , _____________)

B) (_______ , _______ , _______)

C) ________________________________

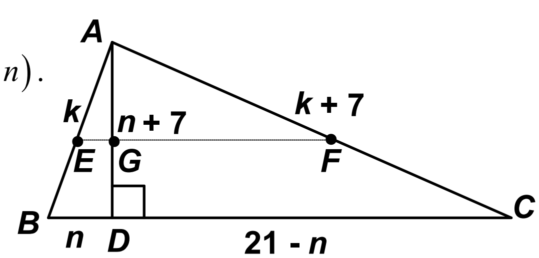
A) Consider all possible connected configurations in the same plane of 4 unit squares (except ), where every two adjacent squares are joined along a common side. Compute the ordered pair (M, m), where M and m denote the maximum and minimum length of a segment which connects two vertices from different squares, and passes through the interiors of at least two squares.



B) O is the center of two concentric circles.
As indicated in the diagram at the right, the degree measures of arcs $\overset{\frown}{AB}$, $\overset{\frown}{DC}$, $\overset{\frown}{BC}$, $\overset{\frown}{AD}$, $\overset{\frown}{FIG}$, $\overset{\frown}{HE}$, $\overset{\frown}{GH}$, and $\overset{\frown}{EF}$ are $5x-1$, $4x-1$, a, b, c, d, 41, and 65, respectively.
If $a-b=62°$ and $c-d=122°$, compute the ordered triple (x, a, c).

C) $\overline{AD}$ is an altitude in ΔABC.
E and F are midpoints of $\overline{AB}$ and $\overline{AC}$, respectively.
For integers n and k, $(AC,\ AD,\ DC) = (k+7,\ n+7,\ 21-n)$.
Compute $\dfrac{|AGE|}{|GFCD|}$, as a simplified fraction.

The notation $|PQR|$ denotes the area of ΔPQR.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2020 SOLUTION KEY

Round 5

A) The segment lengths:

Figure #1: $\sqrt{5},\ \sqrt{13}$

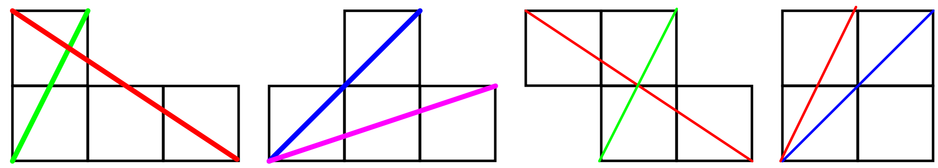
Figure #2:

$2\sqrt{2}=\sqrt{8},\ \sqrt{10}$

Figure #3: $\sqrt{5},\ \sqrt{13}$

Figure #4: $\sqrt{5},\ 2\sqrt{2}$

Thus, $(M,m)=\underline{\left(\mathbf{\sqrt{13},\sqrt{5}}\right)}$.



(1) (2) (3) (4)

B) In the larger circle, we have $m\angle APB=\dfrac{(5x-1)+(4x-1)}{2}=\dfrac{9x-2}{2}$.

In the smaller circle, we have $m\angle EPB=\dfrac{(41)+(65)}{2}=53$.

Equating, $\dfrac{9x-2}{2}=53\Rightarrow x=\dfrac{106+2}{9}=12$.

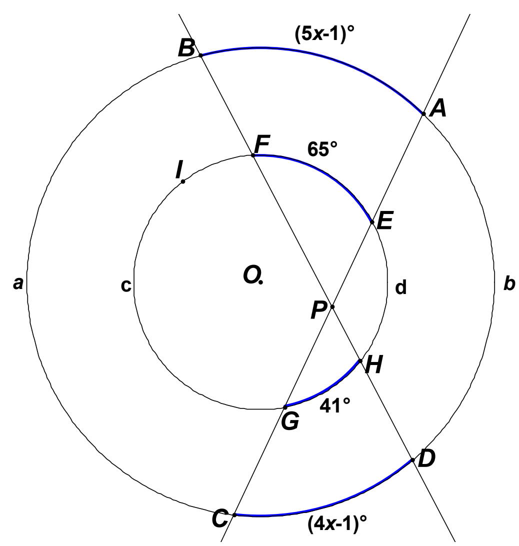
In the larger circle, we have

$$\left.\begin{array}{l}a+b=360-(59+47)=254\\ a-b=62\end{array}\right\}\Rightarrow(a,b)=(158,96).$$

In the smaller circle, we have

$$\left.\begin{array}{l}c+d=360-(65+41)=254\\ c-d=122\end{array}\right\}\Rightarrow(c,d)=(188,66)$$

$(x,a,c)=\underline{\left(\mathbf{12,158,188}\right)}$.



C) E, F = midpoints $\Rightarrow G$ is also a midpoint.

Using the Pythagorean Theorem in ΔBAD and ΔCAD,

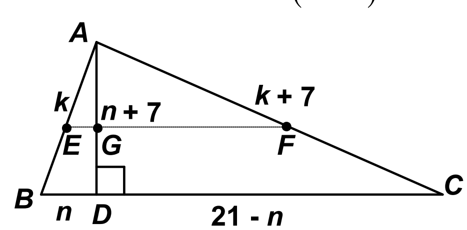
$\Delta BAD: k^2-n^2=(n+7)^2 \qquad \Delta CAD:(k+7)^2-(21-n)^2=k^2+14k+49-441+42n-n^2=(n+7)^2$

$\Rightarrow\ 0=14k+49-441+42n$.

$\Rightarrow\ 0=14k-392+42n\Rightarrow k=28-3n$

$\Rightarrow\ (28-3n)^2=n^2+(n+7)^2\Rightarrow 784-168n+9n^2=2n^2+14n+49$

$\Rightarrow\ 7n^2-182n+735=0\Leftrightarrow n^2-26n+105=(n-5)(n-21)=0$



$$\Rightarrow n=5\Rightarrow k=13\Rightarrow\frac{|AGE|}{|GFCD|}=\frac{\frac{1}{2}\cdot\frac{5}{2}\cdot 6}{\frac{1}{2}\cdot 6\cdot(8+16)}=\underline{\mathbf{\frac{5}{48}}}.$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2020
ROUND 6 ALGEBRA 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) _______________________________

B) _______________________________

C) (_____________ , _____________)

A) Compute the probability (to the nearest tenth of a percent) that an arbitrarily selected base-9 digit is either a multiple of 4, a factor of 6, or an odd prime.

B) The probability that a basketball player makes a 3-point shot is 3/10.
In a 3-point shooting contest, a player must make at least 5 of 10 such shots to win.
If a player has made 3 of 7 shots so far, compute the probability that he will win.

C) The expansion of $\left(3x^2+\frac{2}{x}\right)^k$, where k is a positive integer, contains a constant term N.
Let A be the value of k for which N is the smallest possible nonzero multiple of 7. Compute the ordered pair $\left(A,\frac{N}{7}\right)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2020 SOLUTION KEY

Round 6

A) The base 9 digits are 0, 1, … 8.
Multiples of 4 are 0, 4 and 8. Factors of 6 are 1, 2, 3 and 6. Odd primes are 3, 5 and 7.
Thus, the required probability is **100%**.

B) With 3 shots remaining, he can make either 2 out of 3 shots or all 3 shots to win.

$$\binom{3}{2}\left(\frac{3}{10}\right)^2\left(\frac{7}{10}\right)+\binom{3}{3}\left(\frac{3}{10}\right)^3=\frac{189}{1000}+\frac{27}{1000}=\frac{216}{1000}=\mathbf{0.216}\text{ or }\mathbf{\frac{27}{125}}.$$

C) The general term, i.e., the n^{th} term in the expansion of $\left(3x^2+\frac{2}{x}\right)^k$, is

$$\binom{k}{n}\left(3x^2\right)^{k-n}\left(2x^{-1}\right)^n=\binom{k}{n}\cdot 3^{k-n}\cdot 2^n\cdot x^{2k-3n}$$

If the expansion is to contain a constant term, $2k-3n$ must be zero, implying $n=\frac{2}{3}k$ and since n must be an integer, k must be a multiple of 3. As k increases, so does n.
The smallest value of k will give the smallest constant.

$$k=3\Rightarrow n=2\Rightarrow N=\binom{3}{2}\cdot 3^{3-2}\cdot 2^2\cdot=3\cdot 3^1\cdot 2^2\text{ (not a multiple of 7)}$$

$$k=6\Rightarrow n=4\Rightarrow N=\binom{6}{4}\cdot 3^2\cdot 2^4\text{ (not a multiple of 7)}$$

$$k=9\Rightarrow n=6\Rightarrow N=\binom{9}{6}\cdot 3^3\cdot 2^6=\frac{\cancel{9\cdot 8}^{12}\cdot 7}{\cancel{1\cdot 2\cdot 3}}\cdot 3^3\cdot 2^6\Rightarrow\frac{N}{7}=3^4 2^8=81\cdot 256=20736$$

Thus, (A, N) = **(9, 20736)**.

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 7 TEAM QUESTIONS

ANSWERS

A) ______________________ D) (_______ , _______ , _______)

B) ______________________ E) ___________ : ____________

C) ______________________ F) ______________________

A) Compute <u>all</u> values of k for which there are an infinite number of ordered triples (x, y, z) satisfying

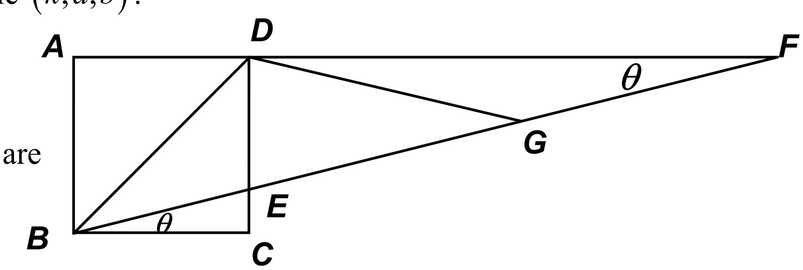
$$\begin{cases} (1) \quad x+2y-z=0 \\ (2) \quad 3x-y+2z=0 \\ (3) \quad x-12y+kz=0 \end{cases}.$$

B) Compute <u>all</u> possible values of x over the reals for which $1+x+x^2+x^3+x^4+....=\sqrt{20x+9}$.
Hint: There is a rational root which is <u>not</u> a unit fraction.

C) The period of $f_1(x)=\sin\left(\frac{x}{92}\right)+\cos\left(\frac{x}{73}\right)$ is $k\pi$.
The period of $f_2(x)=\sin\left(\frac{5x}{364}\right)+\cos\left(\frac{11x}{336}\right)$ is $j\pi$. Compute $j-k$.

D) Let S be 3 more than the reciprocal of some positive integer n.
Let D be the positive difference of 35 and the opposite of this integer.
$D=kS$, for some <u>maximum</u> integer $k \le 10$.
There are exactly two values of n, namely a and b, where $a < b$, which satisfy these conditions. Compute the ordered triple (k,a,b).

E) Given a <u>unit</u> square $ABCD$,
$BD = EG = GF$,
$\theta = 15°$, A, D, and F are collinear, as are B, E, G, and F.
Compute the ratio $CE : DE$.



F) Given: $A=\{(x,y):|2x+y|\le 6\}$, $B=\{(x,y):|2y|+|x|\le 6\}$
Compute the probability that a point selected from region B will be in region A.

MASSACHUSETTS MATHEMATICS LEAGUE
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Team Round

$$\begin{cases} (1) \quad x+2y-z=0 \\ (2) \quad 3x-y+2z=0 \\ (3) \quad x-12y+kz=0 \end{cases}$$

A) Solving the first equation for *x*, we have (4) $x = z - 2y$.
Substituting in the second equation, we have (5)

$$3(z - 2y) - y + 2z = 0 \Rightarrow \boxed{z = \frac{7}{5}y}$$

Substituting for *z* in (4), we have $x = \frac{7}{5}y - 2y \Rightarrow \boxed{x = -\frac{3}{5}y}$.

Finally, substituting for *x* and *z* in the third equation,

$$x-12y+kz=0 \Leftrightarrow -\frac{3}{5}y-12y+k\left(\frac{7}{5}y\right)=0 \Leftrightarrow -3y-60y+7ky=(7k-63)y=0$$

If $y=0$, then $x = z = 0$ and the only solution would be the trivial one - (0, 0, 0).
Thus, $7k-63=0 \Rightarrow k = \underline{\mathbf{9}}$.

Team Round - continued

B) The LHS of the equation is a geometric sequence which *converges* to a unique value $\frac{a}{1-r}$ if and only if $|r|<1$, where r is the common multiplier. Therefore, $\frac{a}{1-r}=\frac{1}{1-x}=\sqrt{20x+9}$.
Squaring both sides and cross-multiplying, $1=(1-x)^2(20x+9) \Leftrightarrow 1=(1-2x+x^2)(20x+9)$
$\Leftrightarrow 1=(20x-40x^2+20x^3)+(9-18x+9x^2) \Leftrightarrow 20x^3-31x^2+2x+8=0$.
The only possible rational roots must be of the form $\pm\frac{A}{B}$, where A is a factor of the constant term 8, and B is a factor of the lead coefficient 20. A: 1,2,4,8 B: 1,2,4,5,10,20
Thus, the possibilities are: $\pm\left(\cancel{1},\cancel{2},\cancel{4},\cancel{8},\cancel{\frac{1}{2}},\cancel{\frac{1}{4}},\cancel{\frac{1}{5}},\cancel{\frac{1}{10}},\cancel{\frac{1}{20}},\frac{2}{5},\frac{4}{5},\frac{8}{5}\right)$, but we were given that the rational root was not a unit fraction, leaving only 6 possibilities to check.

$$\begin{array}{r|rrrr} & 20 & -31 & 2 & 8 \\ & & 16 & -12 & -8 \\ \hline \frac{4}{5} & 20 & -15 & -10 & 0 \end{array}$$

By synthetic division, we confirm that the rational root is $\frac{4}{5}$, and the cubic polynomial factors as
$\left(x-\frac{4}{5}\right)(20x^2-15x-10)=0 \Leftrightarrow (5x-4)(4x^2-3x-2)=0$.

Using the quadratic formula, we find the irrational roots to be $\frac{3\pm\sqrt{9+32}}{8}=\frac{3\pm\sqrt{41}}{8}$.

Clearly, $\frac{3+\sqrt{41}}{8}>\frac{3+6}{8}>1$ and is extraneous. $\left|\frac{3-\sqrt{41}}{8}\right|<1$, so the LHS converges to a unique number. Clearly, $x=\frac{3-\sqrt{41}}{8}<0$ and $\frac{1}{1-x}>0$. Both sides of the original equation were positive, so no extraneous solution was introduced by squaring both sides. Lastly, we must verify that the radicand on the RHS is positive.

$20x+9=\cancel{20}^{5}\left(\frac{3-\sqrt{41}}{\cancel{8}^{2}}\right)+9=\frac{15-5\sqrt{41}+18}{2}=\frac{33-5\sqrt{41}}{2}$ If the numerator is positive, we are home free. We know that $33>5\sqrt{41} \Leftrightarrow 33^2>25\cdot 41 \Leftrightarrow 1089>1025$. Since these inequalities are equivalent and the last inequality is certainly true, the radicand $20x+9$ is positive. Thus, both $\underline{\mathbf{\frac{4}{5},\frac{3-\sqrt{41}}{8}}}$ are roots of the original equation.

Team Round - continued

C) Since the basic sine and cosine functions have periods of 2π, $g(x)=\sin\left(\frac{x}{92}\right)$ completes one cycle over intervals of length 184π; whereas, $h(x)=\cos\left(\frac{x}{73}\right)$ completes one cycle over intervals of length 146π. Since 92 and 73 have no common factors (other than 1), the first function $y=f_1(x)$ completes one cycle over intervals of length $(92\cdot 73)\pi = 6716\pi$, as $y=g(x)$ completes 73 cycles and $y=h(x)$ completes 92 cycles, and no shorter interval contains an integer number of cycles of each individual function.

The components of the second function have periods of $\frac{2\pi}{5/364}=\frac{728\pi}{5}$ and $\frac{2\pi}{11/336}=\frac{672\pi}{11}$

What is the smallest number *N* into which both values will divide?

That is, for what smallest possible *N*-value will both $\frac{5N}{728\pi}=\frac{5N}{2^3\cdot 7\cdot 13\pi}$ and $\frac{11N}{672\pi}=\frac{11N}{2^5\cdot 3\cdot 7\pi}$ be integers? The smallest such value is *product* of the denominators *divided by* the GCF of the denominators, i.e.,

$$\frac{\left(\cancel{2^3}\cdot\cancel{7}\cdot 13\cancel{\pi}\right)\left(2^5\cdot 3\cdot 7\pi\right)}{\cancel{2^3}\cdot\cancel{7}\cancel{\pi}}=32\cdot 3\cdot 7\cdot 13\pi = 8736\pi .$$

The individual functions complete 12 cycles and 13 cycles, respectively.

Thus, $j-k=8736-6716=$ <u>**2020**</u>.

<u>**FYI**</u>**:**

In general, what is the period of $\sin\left(\frac{ax}{b}+h\right)+\cos\left(\frac{cx}{d}+k\right)$, where *a*, *b*, *c*, and *d* are integers sharing no common factors? The answer is $(2bd)\pi$. **The constants *h* and *k* are irrelevant**, since they just determine a reference point (other than the origin) where a cycle begins. The period of the sine function is $\frac{2\pi}{a/b}=\frac{2b\pi}{a}$ and the period of the cosine function is $\frac{2\pi}{c/d}=\frac{2d\pi}{c}$. We must find *integers m* and *n*, so that, after *m* cycles of the sine, and *n* cycles of the cosine, both functions are simultaneously restarting their cycles.

Team Round - continued

We require that $\left(\frac{2b\pi}{a}\right)m = \left(\frac{2d\pi}{c}\right)n$. Canceling the common factors,

$\frac{bm}{a} = \frac{dn}{c} \Leftrightarrow m(bc) = n(ad)$. Since *a*, *b*, *c*, and *d* share no common factors, we must take *m* to be *ad* and *n* to be *bc*. Thus, the sine function has a period of $\frac{2b\pi}{a}$ and goes through *m* = *ad* periods, while the cosine function has a period of $\frac{2d\pi}{c}$ and goes through *n* = *bc* periods.

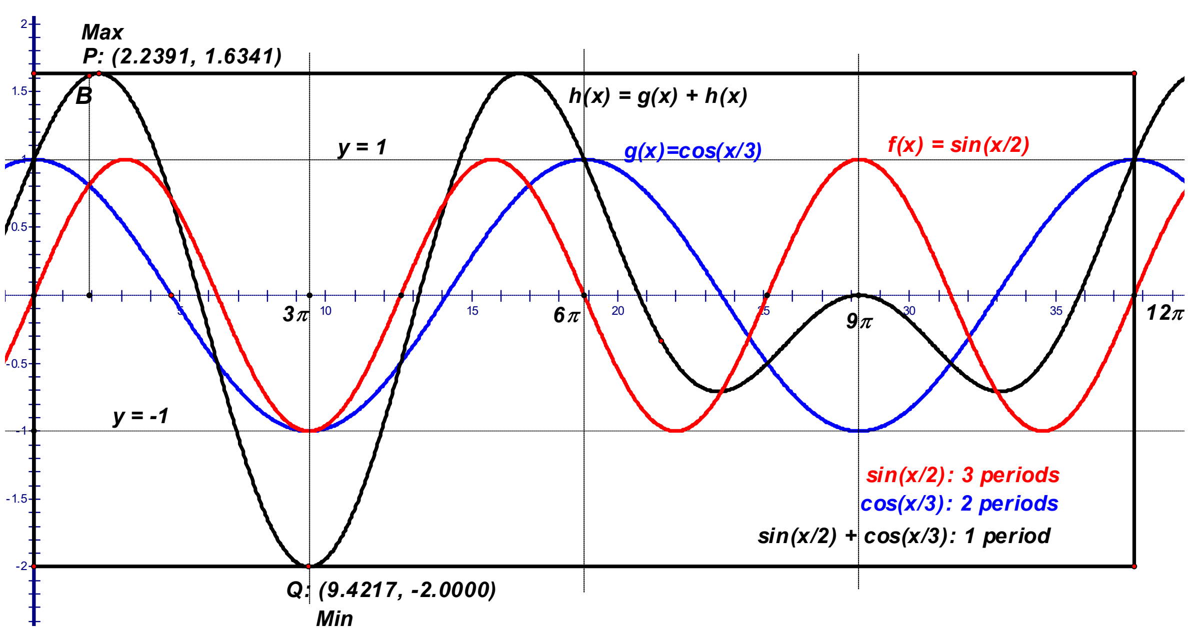
$\left(\frac{2b\pi}{a}\right)m = \left(\frac{2d\pi}{c}\right)n = \left(\frac{2b\pi}{a}\right)ad = \left(\frac{2d\pi}{c}\right)bc = 2bd\pi$ and both functions are returning to their starting position after $2bd\pi$. Therefore, **the constants *a* and *c* are also irrelevant!**
Let's look at a graph using $b = 2$ and $d = 3$, and see that the period is 12π.

$y = \sin\left(\frac{x}{2}\right)$ has competed 3 periods over the 12π interval. Trace over the red graph.

$y = \cos\left(\frac{x}{3}\right)$ has competed 2 periods over the 12π interval. Trace over the blue graph.

The sum of these two functions has completed exactly one cycle. Notice that the minimum occurs where you'd expect it, namely, at $x = 3\pi$ when both $\sin\left(\frac{x}{2}\right)$ and $\cos\left(\frac{x}{3}\right)$ have each reached their minimum values of -1. The maximum value is a story for another day.

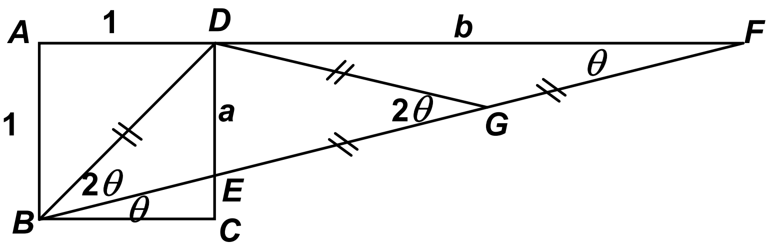
The maximum does <u>not</u> occur at point *B* directly above the point where $y = \sin\left(\frac{x}{2}\right)$ and $y = \cos\left(\frac{x}{3}\right)$ intersect. How come? How would you find the maximum value?



Team Round - continued

D) Given: S is 3 more than the reciprocal of some positive integer n.
$D = kS$ is the positive difference of 35 and the opposite of this integer.
$D = kS \Rightarrow k\left(3+\frac{1}{n}\right) = 35-\left({}^{-}n\right) \Leftrightarrow 3k+\frac{k}{n} = 35+n$
$n^2+(35-3k)n-k=0$
Applying the quadratic formula, $n = \dfrac{(3k-35)\pm\sqrt{(35-3k)^2+4k}}{2}$.
We require the radicand to be a perfect square for some maximum integer $1 \le k \le 10$.
$k=10 \Rightarrow 25+40=65$ rejected
$k=9 \Rightarrow 8^2+36=100$ Bingo!
$n = \dfrac{27-35\pm10}{2} = \dfrac{-8\pm10}{2} = 1,-9 \Rightarrow (k,a,b) = \underline{\mathbf{(9,-9,1)}}$

E) Since G is the midpoint of the hypotenuse of right triangle FED, G is equidistant from E, F, and D.
[It is the center of the circle circumscribed about ΔDEF.]
Thus,



$BD = EG = GF = GD = \sqrt{2}$.
Let $DE = a$, $DF = b$.
$\Delta FED \sim \Delta BEC \Rightarrow \dfrac{CE}{DE} = \dfrac{CB}{DF} \Leftrightarrow \dfrac{1-a}{a} = \dfrac{1}{b}$.
In right ΔFED, $\dfrac{b}{2\sqrt{2}} = \cos 15° = \sin 75° = \dfrac{\sqrt{6}+\sqrt{2}}{4} \Rightarrow b = 2\sqrt{2}\left(\dfrac{\sqrt{6}+\sqrt{2}}{4}\right) = \sqrt{3}+1$.
Therefore, $\dfrac{CE}{DE} = \dfrac{1}{\sqrt{3}+1} = \underline{\dfrac{\mathbf{\sqrt{3}-1}}{\mathbf{2}}}$.
FYI: Notice that $\overrightarrow{BE}$ is <u>one</u> of the angle <u>*trisectors*</u> of $\angle DBC$.
This construction can be used to <u>trisect</u> any acute angle.
Caveat: A Euclidean construction allows the use of an *unmarked* straightedge only.
This construction is possible only if we are permitted to mark two points E' and F' on the straightedge so that $E'F' = 2\cdot BD$, and then align the straightedge so it passes through B.
E' coincides with E (on $\overline{CD}$), and F' coincides with F (on the ray $\overrightarrow{AD}$).
<u>Trisecting a general angle</u>, <u>squaring the circle</u>, and <u>duplicating the cube</u> are the three classical ***impossible*** constructions. The impossibility proofs are quite interesting, but involve some heavy-duty mathematics.

Team Round - continued

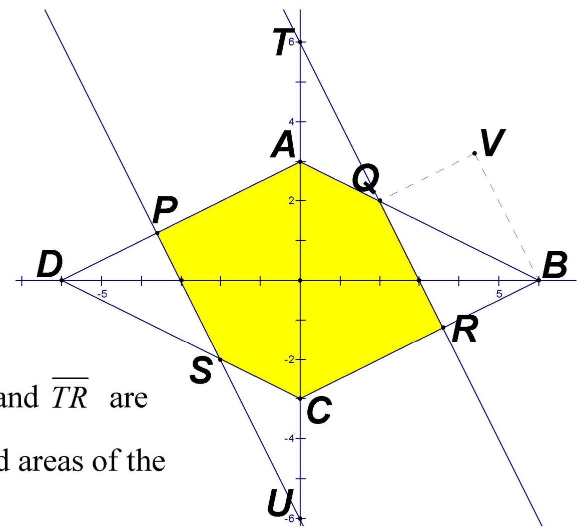
F) Given: $A = \{(x, y) : |2x + y| \le 6\}$, $B = \{(x, y) : |2y| + |x| \le 6\}$

A is a region between the parallel lines $y = -2x - 6$ and $y = -2x + 6$.

B is a region inside the rhombus whose vertices are (0, 3), (6, 0), (0, –3) and (–6, 0).

The required probability is the ratio of the shaded region to the area of the rhombus $ABCD$.

The area of the rhombus is $\frac{1}{2}d_1 d_2 = \frac{1}{2} \cdot 12 \cdot 6 = 36$.



Q is the intersection of $\begin{cases} \overleftrightarrow{TR} : y = -2x + 6 \\ \overleftrightarrow{AB} : x + 2y = 6 \end{cases} \Rightarrow Q(2,2)$.

R is the intersection of

$\begin{cases} \overleftrightarrow{TR} : y = -2x + 6 \\ \overleftrightarrow{BC} : x - 2y = 6 \end{cases} \Rightarrow R(3.6, -1.2)$.

Note: Since $\Delta DPS \cong \Delta BRQ$, and the slopes of $\overline{BC}$ and $\overline{TR}$ are $\frac{1}{2}$ and –2, respectively, $\overline{BC} \perp \overline{TR}$, and the combined areas of the two triangles is equal to the area of rectangle $QRBV$.

$QR = \sqrt{(3.6-2)^2 + (-1.2-2)^2} = \sqrt{1.6^2 + 3.2^2} = 1.6\sqrt{5}$.

$RB = \sqrt{(6-3.6)^2 + (0+1.2)^2} = \sqrt{2.4^2 + 1.2^2} = 1.2\sqrt{5}$.

$\Rightarrow \text{area}(QRBV) = 1.6 \cdot 1.2 \cdot 5 = 8 \cdot 1.2 = 9.6$.

Therefore, the required probability is $\frac{36 - 9.6}{36} = \frac{26.4}{36} = \frac{264}{360} = \frac{22}{30} = \mathbf{\frac{11}{15}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2020 ANSWERS

Round 1 Algebra 2: Simultaneous Equations and Determinants

A) $(5,2)$ B) $(10,23)$ C) $(5,2,1),(5,2,-4)$

Round 2 Algebra 1: Exponents and Radicals

A) $\left(6,\frac{5}{2}\right)$ B) 4 C) $\frac{1}{16}$

Round 3 Trigonometry: Anything

A) $\left(\frac{1}{4},\frac{5\pi}{12}\right)$ B) $\left(20,\frac{4}{5}\right)$ C) $\sqrt{3}$

Round 4 Algebra 1: Anything

A) $-\frac{5}{3}$ B) 29 only C) $\frac{45}{2}$ or 22.5

Round 5 Plane Geometry: Anything

A) $\left(\sqrt{13},\sqrt{5}\right)$ B) (12, 158, 188) C) $\frac{5}{48}$

Round 6 Algebra 2: Probability and the Binomial Theorem

A) 100% (or 1) B) $\frac{27}{125}$ (or 0.216) C) (9, 20736)

Team Round

A) 9

B) $\frac{4}{5},\ \frac{3-\sqrt{41}}{8}$
(Both answers required.)

C) 2020

D) $(9,-9,1)$

E) $\frac{\sqrt{3}-1}{2}$

F) $\frac{11}{15}$