MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A) (____________ , ____________)

B) (____________ , ____________)

C) ______________________________

A) Given: $f(x) = 20 - 18x$ and $g(x) = mx + b$
Compute the ordered pair (m, b) for which $f\left(g\left(x\right)\right) = 18 - 20x$.

B) Given: $f(x) = p(x-1) = q(2x) = 10 - 4x$,
Compute the ordered pair of constants (A, B) for which $p(x) + q(x) = Ax + B$.

C) Given: $y = f\left(x\right) = 2x^2 - 4x - 2$
Point V is the vertex of the graph of this quadratic function.
A linear function intersects $y = f\left(x\right)$ at two points P and Q, whose coordinates are of the form $\left(3 - n, 4 + 2n\right)$. Compute $PV + VQ$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020 SOLUTION KEY

Round 1

A) $f\left(g\left(x\right)\right)=20-18\left(mx+b\right)=\left(20-18b\right)-\left(18m\right)x=18-20x$.

Since this must be true for <u>all</u> values of x, we have $\begin{cases}20-18b=18\\18m=20\end{cases}\Rightarrow\left(m,b\right)=\underline{\left(\mathbf{\frac{10}{9},\frac{1}{9}}\right)}$.

B) Since $p(x-1)=10-4x$, we replace x by $x+1$ to get $p(x)$.
Thus, $p(x)=10-4(x+1)=6-4x$.
Similarly, $q(x)=10-4\left(\frac{x}{2}\right)=10-2x$.
Adding, $r(x)=16-6x\Rightarrow(A,B)=\underline{\mathbf{(-6,16)}}$.

C) $2x^2-4x-2=2\left(x^2-2x+1\right)-2-2=2\left(x-1\right)^2-4\Rightarrow V\left(1,-4\right)$.

Substituting in $y=f\left(x\right)$, $2\left(3-n\right)^2-4\left(3-n\right)-2=4+2n$

$\Leftrightarrow 18-12n+2n^2-12+4n-2=4+2n$

$\Leftrightarrow 2n^2-10n=2n\left(n-5\right)=0\Rightarrow n=0,5$.

$n=0\Rightarrow P\left(3,4\right); n=5\Rightarrow P\left(-2,14\right)$.

$PV+VQ=\sqrt{2^2+8^2}+\sqrt{\left(-3\right)^2+\left(18\right)^2}=\underline{\mathbf{2\sqrt{17}+3\sqrt{37}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020
ROUND 2 ARITHMETIC / NUMBER THEORY

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) a, b, and c are three primes.
Two of the primes differ by 27.
Exactly one of the primes leaves a remainder of 3 when divided by 4.
The sum of the three primes is less than or equal to 132.
Compute the largest possible value of a, b, or c.

B) The product of A and B is 66150.
$A : B = 2 : 3$.
Determine the largest prime number that is a factor of A or B, but not both.

C) What octal (base 8) digits do not occur in the octal representation of the hexadecimal (base 16) integer $DC9C$?

Note: In base 16, the representation of the extra digits are given by the letters,
$A = 10_{10}$, $B = 11_{10}$, ..., $F = 15_{10}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020 SOLUTION KEY

Round 2

A) The difference between any two primes is an even number, unless one of the primes is 2. Therefore, two of the primes must be 2 and 29, and the third prime must be less than or equal to 101. 101 is prime, but is rejected, since none of 2, 29, and 101 leave a remainder of 3 when divided by 4.
Neither 97 nor 89 leave a remainder of 3 when divided by 4, but 83 does.
Thus, the largest prime is **83**.
Note: 91 is not prime, since $91 = 13 \cdot 7$.

B) Let $A = 2n$ and $B = 3n$. Then:
$$6n^2 = 66150 \Rightarrow n^2 = 11025 = 25(441) = 5^2 \cdot 21^2 \Rightarrow n = 105 \Rightarrow (A,B) = (210,315)$$
$$\begin{cases} 210 = 10 \cdot 21 = 2 \cdot 3 \cdot 5 \cdot 7 \\ 315 = 5 \cdot 63 = 3^2 \cdot 5 \cdot 7 \end{cases}$$
Thus, the largest non-common prime factor is **2**.

C) Converting each digit to binary, $DC9C = 1101\;1100\;1001\;1100$.
Every digit in base 16 may be represented by a string of four 1s and 0s.
Since every digit in base 8 is represented by a string of three 1s and 0s, we partition the 16-bit binary string representing $DC9C$ into groups of 3, starting on the right.
$$DC9C = 1101\;\;1100\;\;1001\;\;1100 \Rightarrow 1\|101\|110\|010\|011\|100 = 156234_8$$
Thus, the only octal digits which don't occur are **0** and **7**.

Alternate solution #1 (converting powers of 16 into powers of 8):
Converting to base 10, $DC9C = 13 \cdot 16^3 + 12 \cdot 16^2 + 9 \cdot 16 + 12$
$$= 13 \cdot (2 \cdot 8)^3 + 12 \cdot (2 \cdot 8)^2 + 9 \cdot (2 \cdot 8) + (8+4)$$
$$= 13 \cdot 8^4 + 6 \cdot 8 \cdot 8^2 + (8+1) \cdot 2 \cdot 8 + (8+4)$$
$$= (8+5) \cdot 8^4 + 6 \cdot 8^3 + 2 \cdot 8^2 + 2 \cdot 8 + 8 + 4$$
$$= 1 \cdot 8^5 + 5 \cdot 8^4 + 6 \cdot 8^3 + 2 \cdot 8^2 + 3 \cdot 8 + 4$$
$= 156234_8$, and the same results follow.

Alternate solution #2 (convert from base 16 to base 10 and then back to base 8):

$$DC9C_{16} = 13\left(16^3\right) + 12\left(16^2\right) + 9\left(16^1\right) + 12\left(16^0\right) = 53248 + 3072 + 144 + 12 = 56476$$
Here's a shortcut for converting from base 10 to base 8 (A similar strategy works for any base.) Continue to divide by 8 and save the quotients and the remainders. Stop when the quotient is zero. Read the remainders from bottom to top!

8	56476	Rem
	7059	4
	882	3
	110	2
	13	6
	1	5
	0	1

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) (_____________ , _____________)

B) ______________________________

C) ______________________________

A) $\tan(x)+\cot(x)=f(x)\cdot g(x)$, where *f* and *g* are basic trigonometric functions and $f(0)=1$.
Compute the ordered pair $\left(f(x),g(x)\right)$.

B) Given: $2\cos(5\theta)-\sqrt{3}=0$ and $0<\theta<1000°$
How many values of θ satisfy these two conditions?

C) Express $\sin(3x-\pi)$ strictly in terms of $\sin x$, using only rational coefficients.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020 SOLUTION KEY

Round 3

A) $\tan x + \cot x = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = \dfrac{\sin^2 x + \cos^2 x}{\cos x \sin x} = \dfrac{1}{\cos x \sin x} = \dfrac{1}{\cos x} \cdot \dfrac{1}{\sin x} = \sec x \csc x$.

$\sec(0) = 1 \Rightarrow (f(x), g(x)) = \underline{\mathbf{(\sec x, \csc x)}}$.

B) $\cos(5\theta) = \dfrac{\sqrt{3}}{2} \Rightarrow 5\theta = \{\pm 30° + n(360°) \Rightarrow \theta = \pm 6° + n(72°)$

$0 < 6 + 72n < 1000 \Rightarrow -\dfrac{6}{72} < n < \dfrac{994}{72} < 13^{+} \Rightarrow 0 \le n \le 13 \Rightarrow$ 14 values

$0 < -6 + 72n < 1000 \Rightarrow \dfrac{6}{72} < n < \dfrac{1006}{72} < 13^{+} \Rightarrow 1 \le n \le 13 \Rightarrow$ 13 values $\therefore$ Total: **<u>27</u>**

C) Since the sine function is an odd function, $\sin(3x - \pi) = -\sin(\pi - 3x)$.
By reduction formula [$\sin(\pi - \theta) = \sin\theta$], we have $\sin(\pi - 3x) = \sin 3x$
By transitivity, $\sin(3x - \pi) = -\sin 3x$
Using the $\sin(A + B)$ expansion, $-\sin(2x + x) = -(\sin 2x \cos x + \sin x \cos 2x)$
$\Leftrightarrow -((2\sin x \cos x)\cos x + \sin x(1 - 2\sin^2 x))$
$\Leftrightarrow -((2\sin x \cos^2 x) + \sin x - 2\sin^3 x)$
$\Leftrightarrow -((2\sin x(1 - \sin^2 x)) + \sin x - 2\sin^3 x)$
$\Leftrightarrow -(2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x)$
$\Leftrightarrow \underline{\mathbf{4\sin^3 x - 3\sin x}}$ or $\underline{\mathbf{(\sin x)(4\sin^2 x - 3)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020
ROUND 4 ALGEBRA 1: WORD PROBLEMS

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) _________________________ sq. ft.

A) An MML contest packet consists of 6 pages of individual rounds, a one-page team round, a one-page answer sheet without solutions, and several pages of solutions. All pages are printed single-sided. If 60% of the pages are solutions, how many total pages are in this MML contest packet?

B) The invoice for two purchases for my home remodel project, after a 10% discount for paying cash, was \$3870. There is no sales tax in New Hampshire. The cost for each interior replacement door was \$80 and the cost for each replacement window was \$150.
If X doors and Y windows were purchased, compute the ordered pair (X, Y), given that $X + Y = 38$.

C) A rectangular region is be fenced in. On the long sides the fence posts are 4 feet apart. On the short sides the fence posts are 3 feet apart. If there are 26 fence posts available, compute the area (in square feet) of the *largest* region the fence may enclose.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020 SOLUTION KEY

Round 4

A) Let x denote the number of pages in the solution key.
$0.6(8+x) = x \Rightarrow 48+6x = 10x \Rightarrow x = 12 \Rightarrow$ Total pages: $\underline{\mathbf{20}}$

Alternately, let x denote the total number of pages. $8 = \frac{2}{5}x \Rightarrow x = \underline{\mathbf{20}}$.

B) $\begin{cases} X + Y = 38 \\ .9\left(80X + 150Y\right) = 3870 \end{cases}$

$\Rightarrow .9\left(8X + 15\left(38 - X\right)\right) = 387 \Rightarrow .1\left(570 - 7X\right) = 43 \Rightarrow X = \frac{570-430}{7} = 20$

Therefore, $\left(X, Y\right) = \underline{\left(\mathbf{20,18}\right)}$.

C) Assume the short sides have x 3-foot sections and the long sides have y 4-foot sections.
Then:
Since $(x-1)$ interior fence posts create x 3-foot sections on each short side ($(y-1)$ on the long sides), counting the 4 corner posts, the number of fence posts used is
$2(x-1) + 2(y-1) + 4 = 26 \Leftrightarrow x + y = 13$. Thus, the area is $(3x)(4y) = 12xy$.
The maximum area occurs when x and y are as close to equal as possible.
If $(x, y) = (6, 7)$ or vice versa, we have an area of $12 \cdot 42 = \underline{\mathbf{504}}$ square units.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020
ROUND 5 PLANE GEOMETRY: CIRCLES

ANSWERS

A) ________________________________

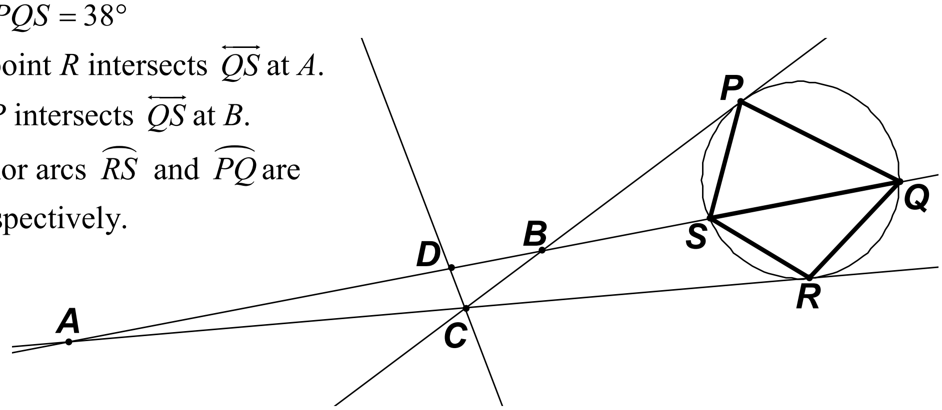
B) ________________________________

C) ________________________________ °

A) A circle has radius of 4 units. A sector of this circle has area 2π units^2.
Compute the perimeter of this sector.

B) Chord $\overline{TR}$ has length 12, subtends a 90° arc in circle O, and is parallel to diameter $\overline{AP}$.
The ratio of the area of trapezoid $TRAP$ to the area of ΔTOP is $k : 1$. Compute k.

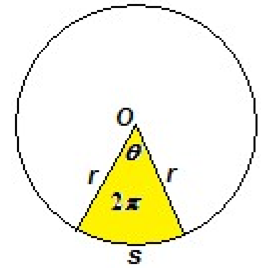
C) Given: $m\angle QSR = 42°$, $m\angle PQS = 38°$
The tangent to the circle at point R intersects $\overleftrightarrow{QS}$ at A.
The tangent to the circle at P intersects $\overleftrightarrow{QS}$ at B.
The degree-measures of minor arcs $\overset{\frown}{RS}$ and $\overset{\frown}{PQ}$ are $(5x-3)°$ and $8(x+1)°$, respectively.
$\overleftrightarrow{CD}$ bisects $\angle ACB$.
Compute $m\angle BDC$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020 SOLUTION KEY

Round 5

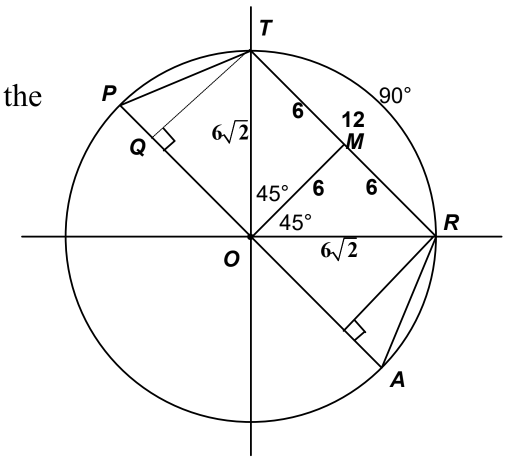
A) Let s denote the arc length of the sector.
Since both the area of a sector and its arc length are proportional to its central angle θ, we have



$$\frac{A_{sector}}{A_{circle}} = \frac{s}{C_{circle}} = \frac{2\pi}{\pi \cdot 4^2} = \frac{s}{2 \cdot \pi \cdot 4} \Leftrightarrow \frac{1}{8} = \frac{s}{8\pi} \Rightarrow s = \pi$$

Thus, the perimeter of the sector is $\underline{\mathbf{8+\pi}}$.

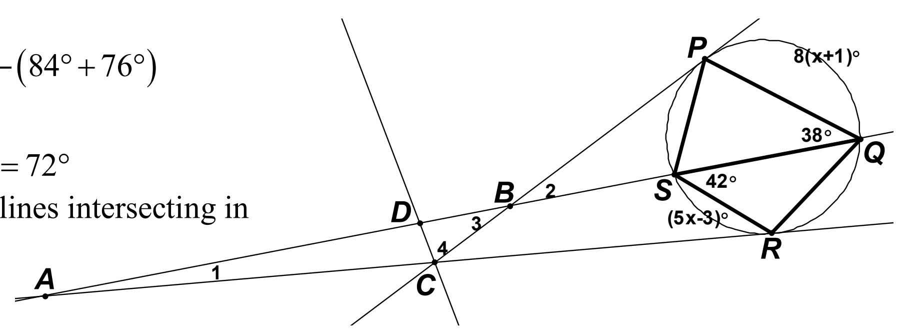
B) It doesn't matter where the 90° arc is drawn, but by drawing the circle centered at the origin, and letting the endpoints of $\overline{TR}$ be located on the axes, we clearly see that the parallel diameter must have length $12\sqrt{2}$ and the height of the trapezoid must be 6, as shown in the diagram at the right.



Thus, the required ratio is
$$\frac{\frac{1}{2}\cdot 6\cdot\left(12+12\sqrt{2}\right)}{\frac{1}{2}\cdot 6\cdot 6\sqrt{2}} = \frac{36\left(\sqrt{2}+1\right)}{18\sqrt{2}}$$
$$= \frac{2\left(\sqrt{2}+1\right)}{\sqrt{2}} = \frac{\sqrt{2}\left(\sqrt{2}+1\right)}{1} = 2+\sqrt{2}.$$

$\Rightarrow k = \underline{\mathbf{2+\sqrt{2}}}$.

C) As the intercepted arcs of inscribed angles,
$\overset{\frown}{QR} = 84°$ and $\overset{\frown}{PS} = 76°$.
$(5x-3)° + 8(x+1)° = 360° - (84° + 76°)$
$\Rightarrow 13x = 360 - 160 - 5 = 195$
$\Rightarrow x = 15 \Rightarrow \overset{\frown}{PQ} = 128°,\ \overset{\frown}{RS} = 72°$
As angles formed by secant lines intersecting in the exterior of the circle,



$$m\angle 1 = \frac{1}{2}(84° - 72°) = 6°$$
$$m\angle 2 = \frac{1}{2}(128° - 76°) = 26°$$
$\Rightarrow m\angle 3 = 26° \Rightarrow m\angle ACB = 148° \Rightarrow m\angle 4 = 74° \Rightarrow m\angle BDC = \underline{\mathbf{80}}°$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Compute the sum of the first six terms of an arithmetic progression in which $(t_1, t_2) = \left(\frac{3}{8}, \frac{17}{24}\right)$.

B) An arithmetic progression (with common difference *d*) and a geometric progression (with common multiplier $r < 1$) have the same first term *a*.
The second term of the arithmetic progression is 12.
The second term of the geometric progression is 8.
If $adr = 16$, compute the number of terms in the arithmetic sequence which must be added to exceed the sum of all the terms in the geometric sequence by 20.

C) Compute $\prod_{k=1}^{k=100} (2020 - 25.25k) + \sum_{k=1}^{k=2020} (-1)^k \cdot (2k+1)$.
Recall: Π is the multiply-a-bunch symbol and Σ is the add-a-bunch symbol.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020 SOLUTION KEY

Round 6

A) The common difference $d = t_2 - t_1 = \frac{17}{24} - \frac{3}{8} = \frac{8}{24} = \frac{1}{3}$.

For an arithmetic progression,

$S_n = \frac{n}{2}\left(2a + (n-1)d\right) \Rightarrow \frac{6}{2}\left(2 \cdot \frac{3}{8} + 5\left(\frac{1}{3}\right)\right) = 3\left(\frac{3}{4} + \frac{5}{3}\right) = \frac{9}{4} + 5 = \underline{\mathbf{\frac{29}{4}}}$ (or equivalent).

B) $\begin{cases} (1)\ a + d = 12 \\ (2)\quad ar = 8 \\ (3)\ \ adr = 16 \end{cases}$

Dividing (3) and (2), $d = 2$.
Substituting in (1), $a = 10$.
Substituting in (2), $r = \frac{4}{5}$.

The sum of all the terms in the geometric sequence is $\frac{a}{1-r} = \frac{10}{1-\frac{4}{5}} = 50$

The arithmetic sequence is 10+12+14+16+18 = 70, and **<u>5</u>** terms produce the required difference.

C) $\prod_{k=1}^{k=100} (2020 - 25.25k) + \sum_{k=1}^{k=2020} (-1)^k \cdot (2k+1)$

Examining the product, $2020 - 25.25k = 20(101) - \frac{101}{4}k = \frac{101(80-k)}{4}$

Thus, the 80th term in the product is 0, implying $\prod_{k=1}^{k=100} (2020 - 25.25k) = 0$.

Examining the summation, for consecutive values of integers k, starting with $k = 1$,
$(-1)^k \cdot (2k+1)$ generates an alternating sequence of odd integers, namely,
$-3 + 5 - 7 + 9 - 11 + 13 + \ldots$.

Specifically, $\sum_{k=1}^{k=2020} (-1)^k \cdot (2k+1)$ generates the sum of 1010 pairs of terms

$(-3+5) + (-7+9) + (-11+13) + \ldots + (-4037 + 4039)$ which is equivalent to $2(1010) = \underline{\mathbf{2020}}$.
Therefore, the grand total is **<u>2020</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____________________________

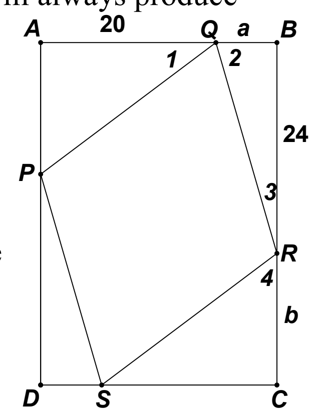
B) (_____________ , ____________) E) ____________________________

C) (_____________ , ____________) F) ____________________________

A) For all integers a and b, $f(a+b) = f(a)f(90-b) + f(90-a)f(b)$, $\begin{cases} f(-18) = 13 \\ f(78) = 3 \\ f(60) = 1 \end{cases}$.

If $b = ka$, for $k \in Z$, then $f(b) = kf(a)$. Compute $f(12)$.

B) Starting with any sequence of k natural numbers, a (single) pair of numbers is replaced by their sum plus their product, and this process is repeated until only one number remains. No matter how the pairings are made, a fixed sequence of natural numbers will always produce the same final number.
Predicting what that final number will be is not easy.
Suppose you start with the sequence 1, a, b, 1, where $a < b$.
Determine the <u>unique</u> ordered pair (a,b) which produces 2019.



C) In the diagram at the right, P, Q, R, and S are connected to form a rhombus with sides of integer length in the interior of a 27 x 39 rectangle $ABCD$. If segments on the sides of the rectangle have lengths as indicated, compute the ordered pair $(\sin(\angle PQR), \sin(\angle QRS))$. Note that the diagram is not necessarily drawn to scale.

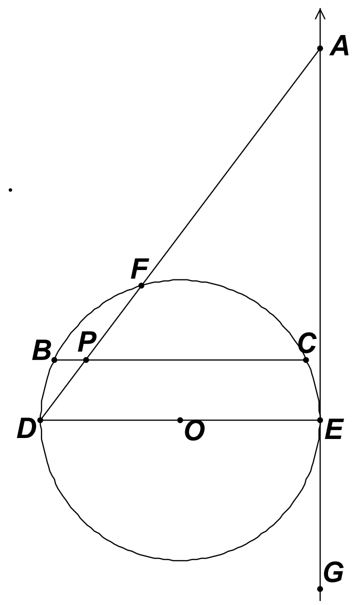
D) The United States post office has issued stamps on almost all the possible 366 days in the year, including February 29th during leap years. The chart at the right specifies the ranges for which data has been collected, but the numbers of days falling within these inclusive ranges have been "blanked" out.

# stamps issued	# days on which this many stamps were issued
0 - 5	x
6 - 10	A
11 - 15	y
16 - 30	B
31 - 40	z
Over 40	C

Suppose $\begin{cases} A = 14 + x \\ B = 4 + y \\ C = z \\ y = x + 32 \\ y = 5C - 7 \end{cases}$.

Specify N, the number of days of the year when no stamps were issued, given that there are 59 days on which 1, 2, 3, 4, or 5 stamps were issued.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020
ROUND 7 TEAM QUESTIONS



E) $\overline{DE}$ is a diameter of circle O.
The area of circle O is 36π.
Line $\overleftrightarrow{AG}$ is tangent to the circle at point E. $DE = .6(AD)$.
If $\dfrac{PF}{PD} = \dfrac{5}{4}$ and $BC = 9.6$, compute $PD + PC$

F) Compute k, if $T = \sum_{n=1}^{n=k} n^3 = 23409$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020 SOLUTION KEY

Team Round

A) Let $(a,b)=(-18,78)$. Then:

$f(-18+78)=f(60)=f(-18)f(90-78)+f(90-(-18))f(78)$.

$1=13\cdot f(12)+f(108)\cdot 3$.

Since $108=12k$, for $k=9$, it follows that $f(108)=9f(12)$.

Substituting, $1=13\cdot f(12)+(9\cdot f(12))\cdot 3=40\cdot f(12)\Rightarrow f(12)=\underline{\boldsymbol{\frac{1}{40}}}$.

Similar problems generated by $\begin{cases}90-a=k(90-b)\\ a+b=60\end{cases}\Rightarrow b=\dfrac{30(3k-1)}{k+1}$

B) $\{1,1,a,b\}\Rightarrow 2019\Leftrightarrow\{3,a,b\}\Rightarrow 2019\Leftrightarrow\{3+4a,b\}\Rightarrow 2019$

$\Leftrightarrow\ 3+4a+4b+4ab=2019$

$\Leftrightarrow\ 3+4(a+b+ab)=2019$

$\Leftrightarrow\ a+b+ab=504$

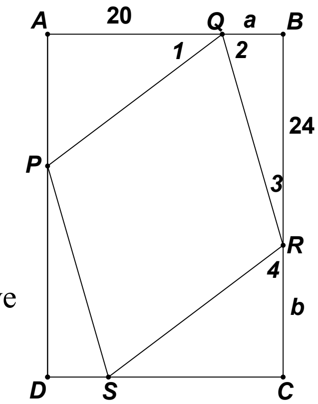
$\Leftrightarrow\ b=\dfrac{504-a}{a+1}$

Clearly, $a=1,2,3$ fail, but then our luck changes!

$a=4\Rightarrow b=\dfrac{504-4}{4+1}=\dfrac{500}{5}=100$ and $(a,b)=\underline{\mathbf{(4,100)}}$.

Team Round - continued

C) Either $\begin{cases} 20+a=27 \\ 24+b=39 \end{cases} \Rightarrow (a,b)=(7,15)$ or $\begin{cases} 20+a=39 \\ 24+b=27 \end{cases} \Rightarrow (a,b)=(19,3)$.



The latter fails to give equal (let alone integer) lengths to the sides of rhombus *PQRS*, but the former gives us two Pythagorean Triples, (7, 24, 25) and 5(3, 4, 5), so each side of *PQRS* has length 25.

Therefore, $\sin\angle 1=\frac{15}{25}=\frac{3}{5}=0.6$, $\sin\angle 2=\frac{24}{25}=0.96$, $\cos\angle 1=0.8$, and $\cos\angle 2=0.28$.

$m\angle PQR=180°-(m\angle 1+m\angle 2)$. By reduction formulas, thinking in terms of radians (or real numbers), i.e., using $\sin(\pi-\theta)=\sin\theta$, we have $\sin\angle PQR=\sin(\pi-(m\angle 1+m\angle 2))=\sin(m\angle 1+m\angle 2)$.

By the sum formulas,

$$\begin{aligned}\sin\angle PQR&=\sin(\angle 1)\cdot\cos(\angle 2)+\sin(\angle 2)\cdot\cos(\angle 1)\\&=0.6\cdot 0.28+0.96\cdot 0.8=0.168+0.768=0.936\end{aligned}.$$

As consecutive angles of a rhombus, $\angle PQR$ and $\angle QRS$ are supplementary, and the sines of supplementary angles are equal.

Therefore, $(\sin(\angle PQR),\ \sin(\angle QRS))=\underline{\mathbf{(0.936,\ 0.936)}}$ or $\underline{\mathbf{\left(\frac{117}{125},\frac{117}{125}\right)}}$.

D) $(x+y+z)+(A+B+C)=366$

$(x+y+z)+(\underline{14+x}+\underline{4+y}+z)=366 \Leftrightarrow 2(x+y+z)=348 \Rightarrow x+y+z=174$.

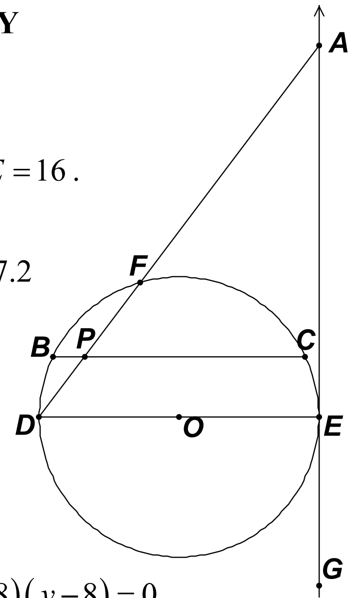
$x=y-32$ and $y=5z-7 \Rightarrow z=\frac{y+7}{5}$. Substituting, $(y-32)+y+\left(\frac{y+7}{5}\right)=174$

$\Rightarrow 5y-160+5y+y+7=870 \Rightarrow 11y=870+153 \Rightarrow y=\frac{1023}{11}=93$

$\Rightarrow (x,A,y,B,z,C)=(61,75,93,97,20,20)$. Since we were given that there were 59 days when 1 to 5 stamps were issued, the number of days when no stamps were issued is $61-59=\underline{\mathbf{2}}$.

<u>FYI</u>: The dates on which no stamps have been issued are 11/8 and 12/25. The date when the most stamps have been issued is 10/1 with 131 stamps issued. These statistics are accurate through the end of 2018. Revised for 2019, the number of face-different stamps issued since 1847 is 6226, including commemorative, definitive, multiple use, semi-postal, airmail, registration, certified, postage due, official, parcel post, and special handling.
Curious about your birthday? Shoot me an email (olson.re@gmail.com).

Team Round - continued



E) $r=6 \Rightarrow DE=12 \Rightarrow AD=20$. Since ADE must be a right triangle, $AE=16$.
Thus, the secant-tangent theorem gives us

$$AE^2=(AF)(AD) \Leftrightarrow 16^2=AF(20) \Rightarrow AF=\frac{256}{20}=\frac{128}{10}=12.8,\ FD=7.2$$

Let $PD=x$. Then:

$$\frac{7.2-x}{x}=\frac{5}{4} \Rightarrow 28.8-4x=5x \Rightarrow x=3.2,\ (PD,PF)=(3.2,4)$$

The Product-Chord theorem guarantees that the product of the lengths of the segments on chord $\overline{BC}$ will equal $4(3.2)=12.8$

Let $PB=y$. Then:

$$y(9.6-y)=12.8 \Leftrightarrow y^2-9.6y+12.8=0 \Leftrightarrow 10y^2-96y+128=2(5y-8)(y-8)=0$$

Thus, $(PB,PC)=(1.6,8)$, and $PD+PC=\underline{\mathbf{11.2}}$.

F) Solving for *k*, given $T=\sum_{n=1}^{n=k} n^3=23409$.

Assuming no knowledge of the formula for the sum of cubes, for small *k*, we look for a pattern:

k		T	4T	$\sqrt{4T}$	Factors
1	1	1	4	2	1·2
2	1+8	9	36	6	2·3
3	1+8+27	36	144	12	3·4
4	1+8+27+64	100	400	20	4·5
…				Pattern:	*k*(*k* + 1)

Thus, the summation formula is $\frac{n^2(n+1)^2}{4} \Rightarrow n^2(n+1)^2=4(23409) \Rightarrow n(n+1)=2\sqrt{23409}$

Since $150^2=22500$ is close to the value of the radicand, we are looking for a 3-digit number close to 150 and ending in a 3 or 7. Testing, $153^2=23409$ BINGO!

Thus, $n(n+1)=2(153)=6(51)=18(17) \Rightarrow k=\underline{\mathbf{17}}$.

The chart above certainly *suggests* the formula for summing successive cubes is $\sum_{k=1}^{k=n} k^3=\frac{n^2(n+1)^2}{4}$. However, it does not constitute a proof. Can you provide an argument that establishes this result in general? One possibility would be to use Mathematical Induction.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2020 ANSWERS

Round 1 Algebra 2: Algebraic Functions

A) $\left(\frac{10}{9}, \frac{1}{9}\right)$ B) $(-6, 16)$ C) $2\sqrt{17}+3\sqrt{37}$

Round 2 Arithmetic/ Number Theory

A) 83 B) 2 C) 0, 7

Round 3 Trig Identities and/or Inverse Functions

A) $(\sec x, \csc x)$ B) 27 C) $4\sin^3 x - 3\sin x$ or $(\sin x)(4\sin^2 x - 3)$

Round 4 Algebra 1: Word Problems

A) 20 B) $(20, 18)$ C) 504

Round 5 Geometry: Circles

A) $8+\pi$ B) $2+\sqrt{2}$ C) $80°$

Round 6 Algebra 2: Sequences and Series

A) $\frac{29}{4}$ (or equivalent) B) 5 C) 2020

Team Round

A) $\frac{1}{40}$

B) $(4, 100)$

C) $(0.936, 0.936)$ or $\left(\frac{117}{125}, \frac{117}{125}\right)$

D) 2 **[**$(x, A, y, B, z, C) = (61, 75, 93, 97, 20, 20)$**]**

E) 11.2

F) 17