**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020**
**ROUND 1 ANALYTIC GEOMETRY: ANYTHING**

**ANSWERS**

A) ( ______________ , ______________ )

B) ( ______________ , ______________ )

C) ( ______________ , ______________ )

A) $\{(x,y): x^2+y^2=A\}$ is inscribed in $\{(x,y): |x|+|y|=B\}$.
Compute the ordered pair $(A,B)$, if the circumference of the circle is $16\pi$.

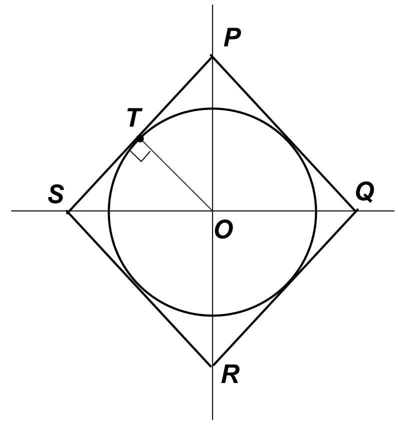
B) The ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the points $M\ (3,1)$ and $N\ (3,-1)$. $M$, $N$, and the points on the ellipse which are on the $y$-axis, are the vertices of an isosceles trapezoid whose area is 9 square units. Compute the ordered pair $(a^2,b^2)$.

C) The equations of the asymptotes of a hyperbola are $x-2y=6$ and $2x+y=7$. One vertex is at $V_1(7,8)$. Compute the coordinates of the second vertex $V_2$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020 SOLUTION KEY**

## Round 1

A) $C = 2\pi r = 16\pi \Rightarrow r = 8 \Rightarrow A = 64$.
$\{(x,y): |x|+|y| = B\}$ is a square with vertices on the coordinate axes. From the diagram at the right, we see that $\Delta TOP$ is an isosceles right triangle whose legs have length 8.
Thus, $B = OP = 8\sqrt{2}$ and $(A,B) = \underline{\mathbf{(64, 8\sqrt{2})}}$.

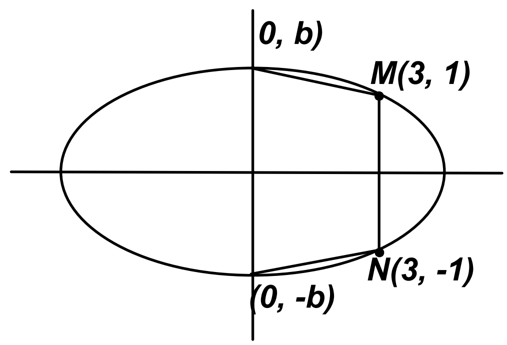



B) $\frac{1}{2}\cdot 3\cdot(2b+2) = 9 \Rightarrow b = 2$.

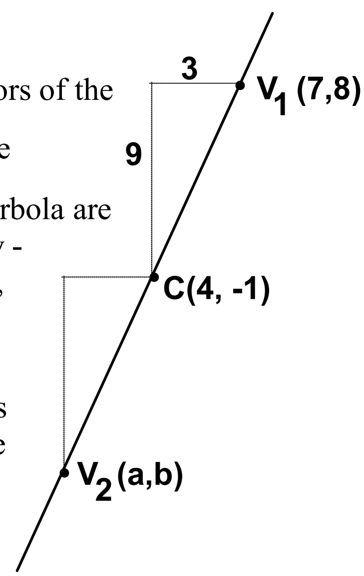
Substituting $(x,y) = (3,1)$ in $\frac{x^2}{a^2}+\frac{y^2}{4} = 1$

$\Rightarrow \frac{9}{a^2} = \frac{3}{4} \Rightarrow a^2 = 12$.

Thus, $(a^2, b^2) = \underline{\mathbf{(12, 4)}}$.




C) Asymptotes: $x-2y=6$ and $2x+y=7$
Symmetry requires that the axes of the hyperbola are the angle bisectors of the opposite angles formed by asymptotes. With slopes of $\frac{1}{2}$ and $-2$, the asymptotes are perpendicular, which implies that the axes of the hyperbola are also perpendicular. This makes finding the equations of the axes easy - we just add and subtract the given equations.** Adding the equations, we have the equation of one of the angle bisectors, $3x - y = 13$.
Subtracting, we have the other, $x+3y=1$.
Since the given vertex satisfies the first equation, this angle bisector is the major axis of the hyperbola. The point of intersection of the angle bisectors (or the two original equations) is the center of the hyperbola, namely $(4,-1)$. From the diagram at the right, the second vertex is $V_2(4-3,-1-9) = \underline{\mathbf{(1, -10)}}$.




** Try proving this on your own before looking at the proof at the end of the solution key. Fortunately, it was **not** necessary to find the equation of the hyperbola.
Its equation is $2x^2 - 3xy - 2y^2 - 19x + 8y + 267 = 0$. If you are familiar with translation and rotation transformations, you might want to try deriving this equation on your own.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020**
**ROUND 2 ALGEBRA 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING**

**ANSWERS**

A) ________________________________

B) ________________________________

C) ________________________________

A) Factor completely over the integers. $x^2(6x-4)+32(2-3x)$

B) $N = 2020^3 - 101^3$ has exactly one two-digit prime factor. Compute the largest three-digit prime factor of *N*.

C) Given: $(x-a)(x-b)(x-c) = x^3 + 2x^2 - 29x + 42 = 0$, and *a*, *b*, and *c* are integers.
If $b = a+1$, compute $a+b+c^3$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020 SOLUTION KEY**

**Round 2**

A) $x^2(6x-4)+32(2-3x) \Leftrightarrow 2\left(x^2(3x-2)-16(3x-2)\right)=2(3x-2)\left(x^2-16\right)$

$\Rightarrow \underline{\mathbf{2(3x-2)(x-4)(x+4)}}$.

B) $2020^3-101^3=(20\cdot 101)^3-101^3=101^3\left(20^3-1\right)=101^3\cdot 7999$

Since 101 is not divisible by 2, 3, 5, or 7, it is a prime number.
(The smallest non-prime not divisible by a prime smaller than 11 is $11^2=121$.)
Thus, 7999 must have a two-digit factor (11, 13, 17 or 19).
Testing each divisor, we find $7999=19\cdot 421$.
Clearly, 421 is not divisible by 2, 3, 5, and 7. Since 11, 13, and 17 failed to be factors of 7999, these primes cannot be factors of 421. If 421 is composite, it must have a factor less than $\sqrt{421}$, but $\sqrt{400}=20<\sqrt{421}<21=\sqrt{441}$. Therefore, I need only check divisibility by 19, since 7999 could have had two factors of 19. Since $421=19\cdot 22+3$, 421 is not divisible by 19, and must be prime. Thus, the largest three-digit factor of *N* is **<u>421</u>**.

Alternately, if you are familiar with the factorization of the difference of perfect cubes $A^3-B^3=(A-B)\left(A^2+AB+B^2\right)$, notice that

$2020^3-101^3=101^3\left(20^3-1^3\right)=101^3\cdot(20-1)\left(20^2+20+1\right)=101^3\cdot 19\cdot 421$, and the same result follows.

C) *a*, *b*, and *c* are roots of the given cubic equation.
Clearly, $abc=-42$.
Since $42=2\cdot 3\cdot 7$, and we are given that two of the roots differ by 1, we check by direct substitution (or synthetic substitution - illustrated below) to see whether any of these values are roots of the equation.

$$\begin{array}{c|cccc} & 1 & 2 & -29 & 42 \\ & & 2 & 8 & -42 \\ \hline 2 & 1 & 4 & -21 & 0 \text{ (Bingo!)} \end{array}$$

Thus, $x=2$ is a root, and the cubic factors to
$(x-2)\left(x^2+4x-21\right)=(x-2)(x+7)(x-3)=0$.
$(a,b,c)=(2,3,-7) \Rightarrow a+b+c^3=2+3-343=\underline{\mathbf{-338}}$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020**
**ROUND 3 TRIG: EQUATIONS WITH A REASONABLE NUMBER OF SOLUTIONS**

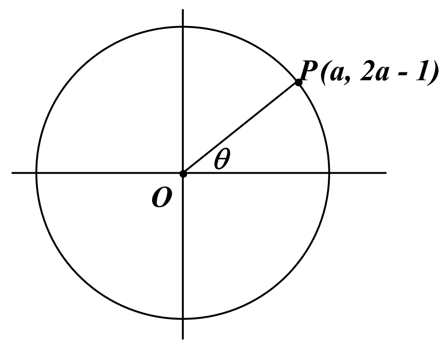
**ANSWERS**

A) ______________________________

B) ______________________________

C) ______________________________

A) Circle $O$ is a unit circle centered at the origin. The coordinates of point $P$ in the first quadrant are $(a,\ 2a-1)$. Compute $\tan\theta$.




B) Compute the sum (in degrees) of the three smallest positive solutions of $1+\sqrt{\sin 6x-1}=\sin 6x$.

C) Given: $A$, $B$ and $C$ are angles in a triangle.

$$\cos A=\frac{13}{15},\ \cos B=\frac{5}{9}.$$

Compute $\cos C$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020 SOLUTION KEY**

**Round 3**

A) $\left(a,\ 2a-1\right)=\left(\cos\theta,\ \sin\theta\right)$

On the unit circle, $x^2+y^2=1 \Leftrightarrow \cos^2\theta+\sin^2\theta=1$

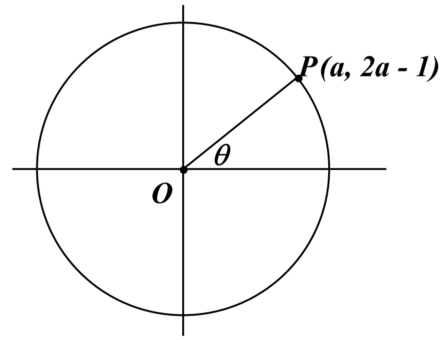
$\Leftrightarrow a^2+\left(2a-1\right)^2=1$

$\Leftrightarrow 5a^2-4a=0$

$\Rightarrow a\left(5a-4\right)=0 \Rightarrow a=\cancel{0},\dfrac{4}{5}$

$\Rightarrow P\left(\dfrac{4}{5},\dfrac{3}{5}\right)$

$\Rightarrow \tan\theta=\dfrac{\sin\theta}{\cos\theta}=\underline{\mathbf{\dfrac{3}{4}}}$.




B) $\sin 6x-1=(\sin 6x-1)^2=\sin^2 6x-2\sin 6x+1\Rightarrow$

$\sin^2 6x-3\sin 6x+2=(\sin 6x-2)(\sin 6x-1)=0$

$\Rightarrow \sin 6x=1 \Rightarrow 6x=90+360n \Rightarrow x=15+60n$ and $n=0,1,2 \Rightarrow 15+75+135=\underline{\mathbf{225}}$.

C) $A+B+C=180° \Rightarrow C=180°-\left(A+B\right)$

$\cos C=\cos\left(180-(A+B)\right)=-\cos\left(A+B\right)=-\cos A\cos B+\sin A\sin B$

$$\begin{cases}\cos A=\dfrac{13}{15}\\ 0<A<180°\end{cases} \Rightarrow \underbrace{0<A<90°}_{\text{quadrant 1}} \Rightarrow \sin A=+\sqrt{1-\left(\dfrac{13}{15}\right)^2}=\sqrt{\dfrac{225-169}{15^2}}=\sqrt{\dfrac{56}{15^2}}=\dfrac{2\sqrt{14}}{15}$$

$$\begin{cases}\cos B=\dfrac{5}{9}\\ 0<A<180°\end{cases} \Rightarrow \underbrace{0<B<90°}_{\text{quadrant 1}} \Rightarrow \sin B=+\sqrt{1-\left(\dfrac{5}{9}\right)^2}=\sqrt{\dfrac{81-25}{9^2}}=\sqrt{\dfrac{56}{9^2}}=\dfrac{2\sqrt{14}}{9}$$

Therefore, $\cos C=-\dfrac{13}{15}\cdot\dfrac{5}{9}+\dfrac{2\sqrt{14}}{15}\cdot\dfrac{2\sqrt{14}}{9}=\dfrac{-65+4\cdot 14}{15\cdot 9}=\underline{\mathbf{-\dfrac{1}{15}}}$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020**
**ROUND 4 ALGEBRA 2: QUADRATIC EQUATIONS**

**ANSWERS**

A) ________________________________

B) ________________________________

C) ________________________________

A) A number like 60 has a lot of divisors (12 to be exact).
Some numbers have considerably more divisors.
Think of some integer *B* which has more divisors than 60.
I'm thinking of the integer *N*.
No matter what number you thought of, my number *N* has more divisors than your number *B*.
Compute <u>all</u> values of *x* for which $x^2 - 3x = 4 - N$.

B) Solve for *x*: $x = 8 - \sqrt{x - \sqrt{x - \sqrt{x - \ldots}}}$

C) Compute <u>all</u> possible values of the constant *a* for which the following equation (in *x*)

$$(a-4)x^2 + (4a+1)x + 4a = 0$$

has exactly one solution.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020 SOLUTION KEY**

**Round 4**

A) *N* must be zero. Zero has an infinite number of divisors, since every number (except 0) divides into 0.
$x^2-3x=4-N \Leftrightarrow x^2-3x-4=0 \Leftrightarrow (x-4)(x+1)=0 \Rightarrow x=\underline{\mathbf{-1,4}}$.

B) Since $\sqrt{x-\sqrt{x-\sqrt{x-...}}}$ must denote a positive number, $x<8$.
$x=8-\sqrt{x-\sqrt{x-\sqrt{x-...}}} \Leftrightarrow \boxed{x-8=-\sqrt{x-\sqrt{x-\sqrt{x-...}}}}=-\sqrt{x+(x-8)}$
Squaring both sides, $x^2-16x+64=2x-8 \Leftrightarrow x^2-18x+72=(x-6)(x-12)=0 \Rightarrow x=6,12$
However, since $x=12$ is extraneous, the only solution is $x=\underline{\mathbf{6}}$.

C) If $a=\underline{\mathbf{4}}$, the equation is linear, namely $17x+16=0$, and there is exactly one solution $x=-\dfrac{16}{17}$.
If $a \neq 4$, the equation is quadratic and to guarantee exactly one solution the discriminant must be zero. Thus, $(4a+1)^2-16a^2+64a=72a+1=0 \Rightarrow a=\underline{\mathbf{-\dfrac{1}{72}}}$.

# MASSACHUSETTS MATHEMATICS LEAGUE
# CONTEST 4 - JANUARY 2020
# ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

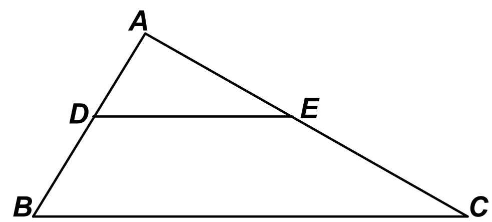
**ANSWERS**

A) ______________________________

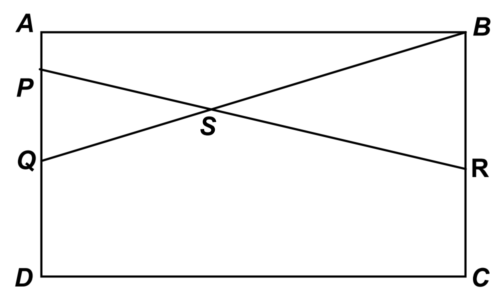
B) ______________________________

C) ______________________________

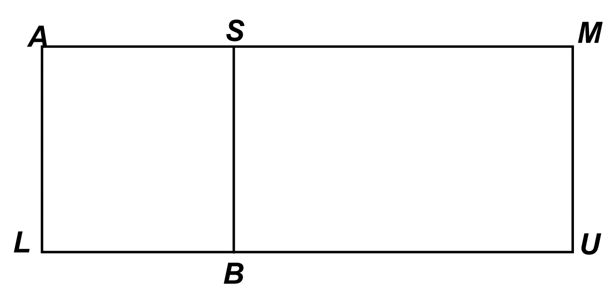
A) Given: $\overline{DE} \parallel \overline{BC}$, $AD = 2.4$, $DB = 4.2$
If the area of *DBCE* is 30, compute the area of $\Delta ADE$.




B) In rectangle *ABCD*, $AP:PQ:QD = 1:2:3$, $BR:RC = 5:4$, $AB:AD = 2:1$, and the perimeter of *ABCD* is 108.
Compute the combined area of $\Delta PQS$ and $\Delta BRS$.

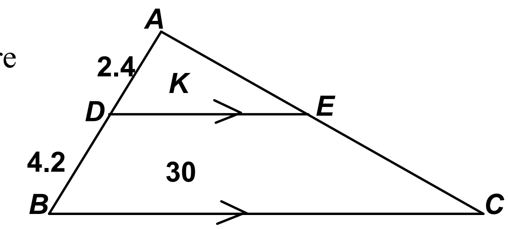



C) Given: Rectangle *ALBS* ~ rectangle *SMUB*
$AM = 12$, $AS < SM$, $MU = n$,
where $n$ is an integer
Compute the largest possible area of *SMUB*.

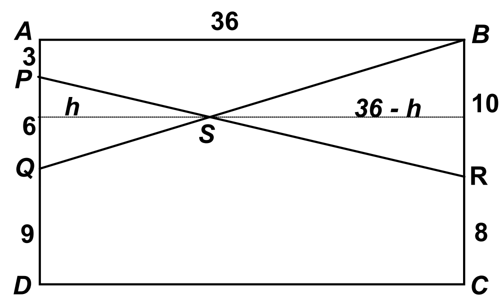


**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020 SOLUTION KEY**

**Round 5**




A) $\overline{DE} \parallel \overline{BC} \Rightarrow \Delta ADE \sim \Delta ABC$. The areas of these triangles are in the ratio of the squares of the sides, namely, $\left(\frac{2.4}{6.6}\right)^2 = \left(\frac{4}{11}\right)^2 = \frac{16}{121}$. Thus, the ratio of the area of $\Delta ADE$ to the area of $DBCE$ is 16 : 105 $\Rightarrow \frac{K}{30} = \frac{16}{105} \Leftrightarrow \frac{K}{2} = \frac{16}{7} \Rightarrow K = \underline{\boldsymbol{\frac{32}{7}}}$.




B) Let the common unit on $\overline{AD}$ be $a$ and the common unit on $\overline{BC}$ be $b$.
$AD = BC \Rightarrow 6a = 9b \Rightarrow b = \frac{2}{3}a$.
$2(6a) + 2(12a) = 36a = 108 \Rightarrow a = 3$ and the lengths are as indicated in the diagram at the right.
Since $\Delta PQS \sim \Delta RBS$, we have
$\frac{h}{36-h} = \frac{6}{10} = \frac{3}{5} \Rightarrow 5h = 108 - 3h \Rightarrow h = 13.5$.
Thus, the required area is $\frac{1}{2}\cdot 6 \cdot 13.5 + \frac{1}{2}\cdot 10 \cdot 22.5 = 40.5 + 112.5 = \underline{\mathbf{153}}$.

C) Let $AS = x$. Then: $\frac{x}{n} = \frac{n}{12-x} \Rightarrow x^2 - 12x + n^2 = 0 \Rightarrow x = \frac{12 \pm \sqrt{4\left(36-n^2\right)}}{2} = 6 \pm \sqrt{36-n^2}$
Since $AS < SM$, $x = 6 - \sqrt{36-n^2}$ and $SM = 6 + \sqrt{36-n^2}$.
Thus, the dimensions of $SMUB$ are $n$ by $\left(6 + \sqrt{36-n^2}\right)$.

As $n$ decreases, so does the value of the area expression $n\left(6 + \sqrt{36-n^2}\right)$. Since $n$ can't be 6 ($n = 6 \Rightarrow AS = SM$), $n = 5$ gives the largest possible area $\underline{\mathbf{5\left(6+\sqrt{11}\right)}}$ or $\underline{\mathbf{30 + 5\sqrt{11}}}$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020**
**ROUND 6 ALGEBRA 1: ANYTHING**

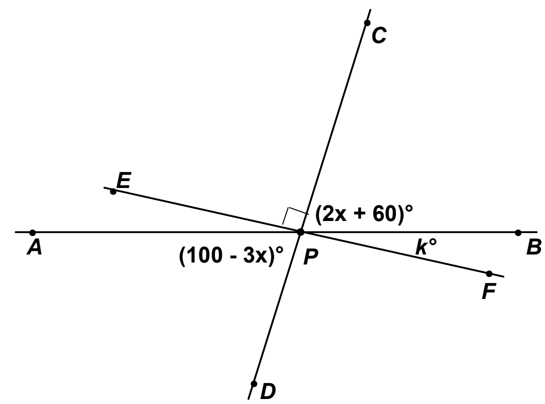
**ANSWERS**

A) ______________________________

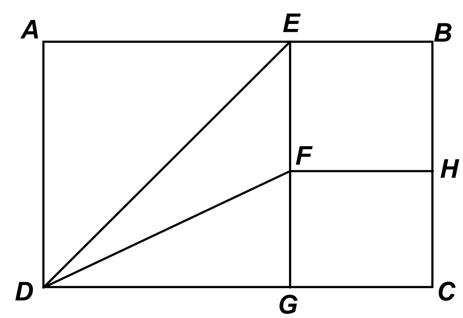
B) ______________________________

C) ______________________________

A) Three lines intersect at point $P$.
Compute $k$, if $\overline{CD} \perp \overline{EF}$ .




B) $ABCD$, $BEGC$ and $HFGC$ are rectangles.
$AEGD$ and $EBHF$ are squares. $BE = 4$, $DF = 20$
Compute the ratio of the perimeter of $EBHF$
to the perimeter of $HFGC$.

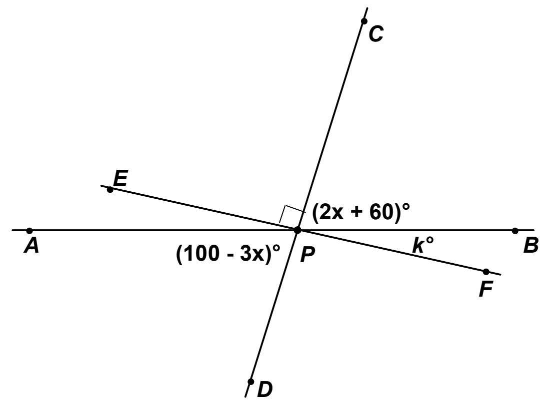


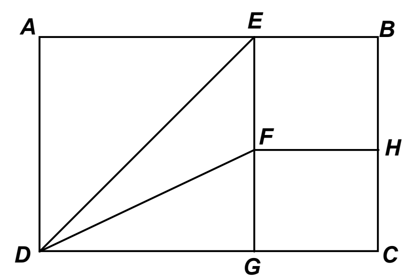

C) Solve for $x$ over the reals: $\dfrac{2x+1}{11-2x} \geq 1$

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020 SOLUTION KEY**

**Round 6**

A) The measures of vertical angles are equal.
$100 - 3x = 2x + 60 \Rightarrow 5x = 40 \Rightarrow x = 8$
$\Rightarrow m\angle CPB = 76°.$
$\overline{CD} \perp \overline{EF} \Rightarrow k + 76 = 90 \Rightarrow k = \underline{\mathbf{14}}$ .




B) If $FG = HC = x$, then the required ratio is 16 : (8 + 2$x$) or $\dfrac{8}{4+x}$.
Using the Pythagorean Theorem on $\Delta FDG$, $(x+4)^2 + x^2 = 20^2$
$\Rightarrow x^2 + 4x - 192 = 0 \Leftrightarrow (x-12)(x+16) = 0$, or
notice that 4(3-4-5) = (12-16-20), and we have a right triangle with hypotenuse of length 20 and legs whose lengths differ by 4.
Either way, $x$ = 12, and the required ratio is $\dfrac{8}{16} = \underline{\dfrac{\mathbf{1}}{\mathbf{2}}}$ .

C) $\dfrac{2x+1}{11-2x} \ge 1 \Leftrightarrow \dfrac{2x+1}{11-2x} - 1 \ge 0 \Leftrightarrow \dfrac{2x+1-11+2x}{11-2x} \ge 0 \Leftrightarrow \dfrac{4x-10}{11-2x} \ge 0$
Critical values are $\dfrac{5}{2}$ and $5\dfrac{1}{2}$. Between these values, both binomials are positive and, therefore, the quotient is positive, as required.
Avoiding division by zero, we have $\underline{\dfrac{\mathbf{5}}{\mathbf{2}} \le \boldsymbol{x} < \dfrac{\mathbf{11}}{\mathbf{2}}}$ (or equivalent).

# MASSACHUSETTS MATHEMATICS LEAGUE
# CONTEST 4 - JANUARY 2020
# ROUND 7 TEAM QUESTIONS

## ANSWERS

A) ( ____ , ____ , ____ , ____ ) D) ____________________________

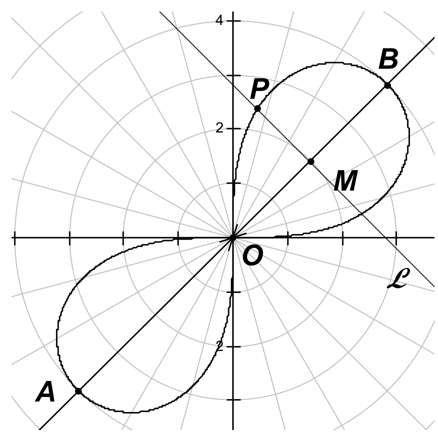
B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

A) $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ is an equation of a horizontal ellipse with axes parallel to the coordinates axes, and passing through the points $A(-1,\ 28), B(-26,\ 4), C(-16,\ -15.2),\ D(14,\ 23.2),\ E(19,\ 18.4)$.
Compute the ordered quadruple $(h,k,a,b)$.

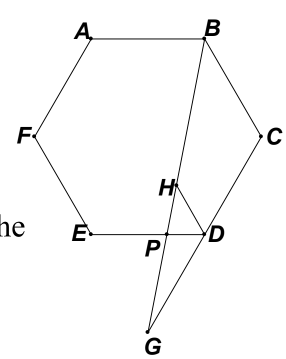
B) The polynomial $x^7+3x^6+x^5-2x^4+x^3+x^2-x$ factors as the product of 4 distinct polynomials in simplest form. These polynomials have 1, 2, 3, and 4 terms respectively, and all have a positive lead coefficient. For $x=m$, the monomial factor equals the quadratic factor. For $x=n$, the cubic factor equals the linear factor. Compute <u>all</u> possible values of $m+n$.

C) The graph of $S=\{(r,\theta)\,|\,r^2=16\sin 2\theta\}$ is symmetric about $\overleftrightarrow{AB}$. $O$ is the origin of the polar coordinate system. $M$ is the midpoint of $\overline{OB}$. $\mathscr{L}$ is perpendicular to $\overleftrightarrow{AB}$ at $M$, and intersects the graph of $S$ at the point $P$, as indicated in the diagram at the right. If $P$ has coordinates $(r_1,\alpha)$, compute $\sin 2\alpha$.




D) Given: $2x^2-16x+k=0$, where $k$ is an integer.
Let $S$ and $P$ denote the sum and product, respectively, of the roots of this quadratic equation.
Compute <u>all</u> values of $S^2+P^2$ which are integer perfect squares.

E) In regular hexagon $ABCDEF$, $AB=4$, and $\overline{CD}$ is extended to $G$ so that $CD=DG$. $\overline{BG}$ intersects $\overline{DE}$ at $P$, $H$ is on $\overline{BG}$ so that $\overline{DH}\parallel\overline{CB}$.
Compute $HP$.




F) $N=A3B4_5$ is a 4-digit base 5 integer which is divisible by 3. Compute the sum of all possible values of $N$, and express the result <u>in base 5</u>.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020 SOLUTION KEY**

**Team Round**

A) Since we were given that the graph was a horizontal ellipse, if the uppermost and leftmost points are *A* and *B*, respectively, then the center would have to be at $(-1,4)$. This is also suggested by the fact that $(-1,4)$ is the midpoint of $\overline{CD}$. Assuming this center point, we'll solve for *a* and *b* and check to see whether all 5 points satisfy the equation.

$$A(-1,28): \frac{(x+1)^2}{a^2}+\frac{(y-4)^2}{b^2}=1 \Rightarrow b^2=24^2 \qquad B(-26,4): \frac{(x+1)^2}{a^2}+\frac{(y-4)^2}{b^2}=1 \Rightarrow a^2=25^2$$

Checking points *C* and *D*, we notice that substituting $x=-16$ or $x=14$ gives us $\frac{15^2}{25^2}=\left(\frac{3}{5}\right)^2=\frac{9}{25}$

which requires that $\frac{(y-4)^2}{576}=1-\frac{9}{25}=\frac{16}{25}$. $y=-15.2$ or $y=23.2 \Rightarrow$

$$\frac{(-19.2)^2}{24^2}=\left(\frac{19.2}{24}\right)^2=\left(\frac{3.2}{4}\right)^2=(0.8)^2=\left(\frac{4}{5}\right)^2=\frac{16}{25}.$$

Bingo - *C*, *D* check.

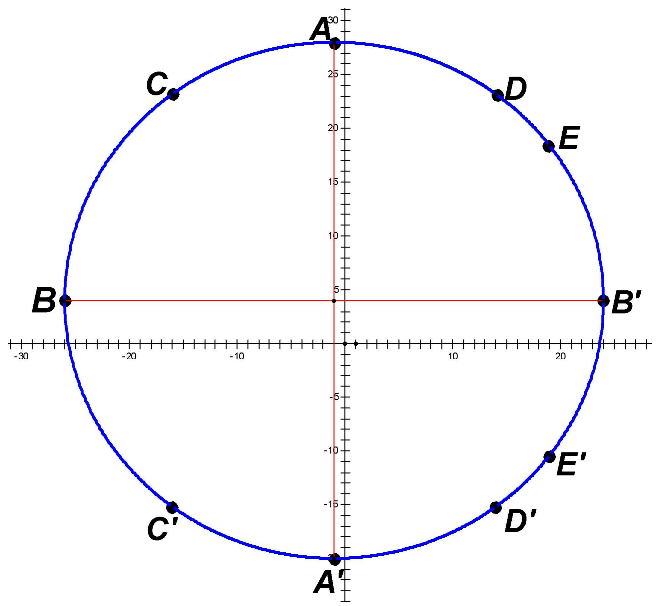
What about $E(19,18.4)$?

$$\frac{(x+1)^2}{25^2}+\frac{(y-4)^2}{24^2}=1$$

$$\Rightarrow \frac{(y-4)^2}{24^2}=1-\left(\frac{20}{25}\right)^2=1-\left(\frac{4}{5}\right)^2=\frac{9}{25}.$$

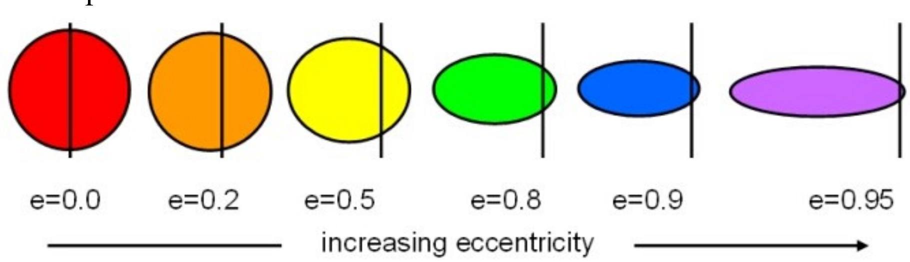
Does $\frac{(18.4-4)^2}{24^2}=\frac{9}{25}$?

$$\frac{(14.4)^2}{24^2}=\left(\frac{14.4}{24}\right)^2=\left(\frac{1.8}{3}\right)^2=(0.6)^2=\left(\frac{3}{5}\right)^2=\frac{9}{25}.$$




*E* checks! Thus, all 5 points satisfy $\frac{(x+1)^2}{25^2}+\frac{(y-4)^2}{24^2}=1$ and $(h,k,a,b)=\underline{\mathbf{(-1,4,25,24)}}$.

**<u>FYI:</u>** $c=7$ gives an eccentricity of $e=\frac{c}{a}=\frac{7}{25}=0.28$. We know that $0<e_{ellipse}<1$, and as the eccentricity *approaches* 0, the ellipse is becoming more and more circular. In each diagram, the vertical line passes through one of the foci on the major axis of the ellipse. The segment in the interior of the ellipse is referred to as a focal chord.



**Team Round - continued**

B) The monomial factor is *x*.

Since the polynomial expression $x^7+3x^6+x^5-2x^4+x^3+x^2-x$ evaluates to zero for $x=-1$, the linear factor is $x+1$.

Dividing synthetically by these two factors, the quotient is $x^5+2x^4-x^3-x^2+2x-1$, which we are given must have quadratic and cubic factors. Clearly, the lead coefficients and constant terms in each must be 1. Therefore, we assume the factors are $\left(x^2+Ax-1\right)$ and $\left(x^3+Bx^2+Cx+1\right)$.

Of course, we realize the constant terms could be reversed. If our assumption is correct, then we can write expressions for the coefficients in terms of *A*, *B*, and *C*. Specifically,

$$\begin{cases} (1)\ x^4:\ B+A=2 \\ (2)\ x^3:\ C+AB-1=-1 \\ (3)\ x^2:\ 1+AC-B=-1 \\ (4)\ x\ :\ A-C=2 \end{cases} \Rightarrow \begin{cases} B=2-A \\ C=A-2 \end{cases}$$

Substituting in (2), $(A-2)+A(2-A)=0 \Leftrightarrow A^2-3A+2=(A-1)(A-2)=0 \Rightarrow A=1,2$

$A=1 \Rightarrow (B,C)=(1,-1)$ and these values check in all 4 equations.

$A=2 \Rightarrow (B,C)=(0,0)$ and these values fail in the third equation.

Therefore, the complete factorization is $x(x+1)\left(x^2+x-1\right)\left(x^3+x^2-x+1\right)$

If we had reversed the original constant terms, the lead coefficients of the quadratic and cubic factors would have been $-1$, which would be rejected, since all lead coefficients were given as positive.

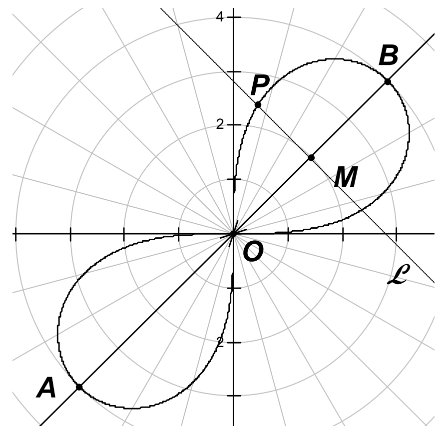
For $x=m, m^2+m-1=m \Rightarrow m=\pm 1$.

For $x=n$, $n^3+n^2-n+1=n+1 \Rightarrow n^3+n^2-2n=n\left(n^2+n-2\right)=0$

$\Rightarrow n(n-1)(n+2)=0 \Rightarrow n=0,1,-2$.

Thus, $m+n=\underline{\mathbf{-3,-1,0,1,2}}$.

**Team Round - continued**




C) Given: $S=\{(r,\theta)\mid r^2=16\sin 2\theta\}$

Since $OB = 4$ and $OM = 2$, the coordinates of midpoint $M$, in rectangular coordinates, are $\left(\sqrt{2},\sqrt{2}\right)$.

The equation of $\mathscr{L}$ is $\left(y-\sqrt{2}\right)=-1\left(x-\sqrt{2}\right)$ or $x+y=2\sqrt{2}$.

Substituting for $x$ and $y$, the polar equation is $r(\cos\theta+\sin\theta)=2\sqrt{2}$ or $r=\dfrac{2\sqrt{2}}{\sin\theta+\cos\theta}$.

Substituting for $r$ in the equation of the polar graph, $\dfrac{8}{\left(\sin\theta+\cos\theta\right)^2}=16\sin 2\theta$.

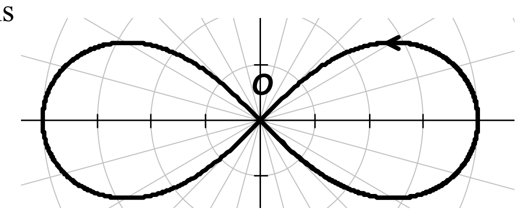
$\Leftrightarrow \dfrac{1}{1+\sin 2\theta}=2\sin 2\theta \Leftrightarrow 2\sin 2\theta(1+\sin 2\theta)=1 \Leftrightarrow 2\sin^2 2\theta+2\sin 2\theta-1=0$

Applying the quadratic formula, $\sin 2\theta=\dfrac{-2\pm\sqrt{4+8}}{4}=\dfrac{-2\pm 2\sqrt{3}}{4}=\dfrac{-1\pm\sqrt{3}}{2}$.

Since $\dfrac{-1-\sqrt{3}}{2}<-1$, it is rejected, and $\sin 2\alpha=$ $\underline{\boldsymbol{\dfrac{\sqrt{3}-1}{2}}}$ .

The graph is referred to as a (rotated) Lemniscate of Bernoulli.
Its general equation is $r^2=a^2\sin 2\theta$, where the constant $a$ denotes the distance from $O$ to $B$.
It's symmetric with respect to the 45° line and the origin.
You should verify that the rectangular equivalent is $\left(x^2+y^2\right)^2=2a^2xy$.
The maximum width occurs along the axis of symmetry.
<u>Challenge</u>: What's the maximum width of a loop (in terms of $a$) measured perpendicular to the axis of symmetry?
A lemniscate is really a two-leaved rose. If the axis of symmetry is along the $x$-axis, the graph is shown in the diagram at the right, and the equations are:

$$\begin{cases}\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)\\ r^2=a^2\cos 2\theta\end{cases}$$




Have fun exploring!

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2020 SOLUTION KEY**

**Team Round - continued**

D) After normalizing the quadratic equation, $x^2 - 8x + \frac{k}{2} = 0$, values (expressions) for $P$ and $S$ can be read directly from the equation. $(P,S) = \left(8, \frac{k}{2}\right) \Rightarrow S^2 + P^2 = 8^2 + \frac{k^2}{4} = \frac{k^2 + 256}{4}$.

We require that, for some integer $j$,

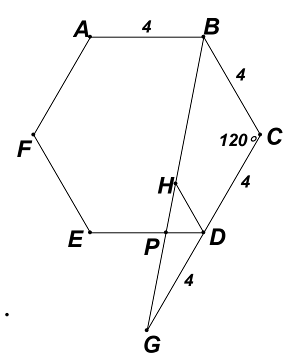
$\frac{k^2+256}{4} = j^2 \Leftrightarrow 4j^2 - k^2 = 256 \Leftrightarrow (2j+k)(2j-k) = 256$.

Since $k$ and $j$ are integers, the left side of the equation is the product of two integers and the only solutions come from factoring 256 as the product of two factors, and setting $2j+k$ equal to the larger factor.

| | | | | | |
|---|---|---|---|---|---|
| $2j+k$ | 256 | 128 | 64 | 32 | 16 |
| $2j-k$ | 1 | 2 | 4 | 8 | 16 $\Rightarrow j = 8,\ 10,\ 17 \Rightarrow S^2 + P^2 =$ **<u>64, 100, 289</u>**. |
| $4j$ | $\cancel{257}$ | $\cancel{130}$ | 68 | 40 | 32 |

Check: $j = 8,\ 10,\ 17 \Rightarrow k = 0, 12, 30$.

$$S^2 + P^2 = \frac{k^2+256}{4} = \frac{\begin{cases}0\\144\\900\end{cases} + 256}{4} = \frac{\begin{cases}256\\400\\1156\end{cases}}{4} = 64,\ 100,\ 289.$$




E) $AB = BC = CD = DG = 4$. Since $\overline{HD} \parallel \overline{BC}$,

$m\angle HDC = m\angle HDE = m\angle EDG = 60°$.

$\Delta GDH \sim \Delta GCB \Rightarrow \frac{HD}{4} = \frac{4}{8} = \frac{1}{2} \Rightarrow HD = 2$

By the Law of Cosines,

$HG^2 = 2^2 + 4^2 - 2 \cdot 2 \cdot 4 \cos 120° = 20 - 16\left(-\frac{1}{2}\right) = 28 \Rightarrow HG = 2\sqrt{7}$.

Since $\overrightarrow{DP}$ bisects $\angle HDG$, by the angle bisector theorem, $HP : PG = HD : DG = 2 : 4 = 1 : 2$.

Let $HP = k$, then $PG = 2k$, so $3k = 2\sqrt{7}$ and $k = HP = \underline{\boldsymbol{\frac{2\sqrt{7}}{3}}}$.

**Team Round - continued**

F) $A3B4_5 = 5^3 \cdot A + 5^2 \cdot 3 + 5 \cdot B + 4 = 79 + 125A + 5B = 3k$

Thus, $\dfrac{79+125A+5B}{3}$ must be an integer.

$$\frac{79+125A+5B}{3} = \frac{78+123A+3B}{3} + \frac{1+2A+2B}{3} = 26 + 41A + B + \frac{1+2(A+B)}{3}$$

$1+2(A+B)$ must be a multiple of 3. A chart will help keep track of the possibilities.

Remember: As base 5 integers, $0 \le A \le 4$ and $0 \le B \le 4$. However, $A \ne 0$.

| 1+2(*A*+*B*) | 2(*A*+*B*) | (*A*, *B*) |
|---|---|---|
| 0 | -1 | None |
| 3 | 2 | ~~(0,1)~~, (1,0) |
| 6 | 5 | None |
| 9 | 8 | ~~(0,4)~~, (4,0), (1,3), (3,1), (2,2) |
| 12 | 11 | None |
| 15 | 14 | (3,4), (4,3) |
| 18 | 17 | None |
| 21 (or more) | 20, 23,... | None |

Thus, there are 7 possible ordered pairs which give us
$1304_5$, $4304_5$, $1334_5$, $3314_5$, $2324_5$, $3344_5$, $4334_5$.

We could convert each number to base 10, add 'em, and convert the base 10 sum back to base 5.
[$204+579+219+459+339+474+594 = 2868 \Rightarrow 42433_5$]

But why not just add them in base 5?

We proceed column by column from right to left, as we would in base 10, but the carry is groups of 5, instead of groups of 10.

| Place Value >> | <u>**125**</u> | <u>**25**</u> | <u>**5**</u> | <u>**1**</u> |
|---|---|---|---|---|
| | 1 | 3 | 0 | 4 |
| | 4 | 3 | 0 | 4 |
| | 3 | 3 | 1 | 4 |
| | 1 | 3 | 3 | 4 |
| | 2 | 3 | 2 | 4 |
| | 3 | 3 | 4 | 4 |
| | 4 | 3 | 3 | 4 |
| Base 10 Totals: | (18) | (21) | (13) | (28) |
| Carry | 4 | 3 | 5 | |
| | 22 | 24 | 18 | |
| 4 | 2 | 4 | 3 | 3 |

Start in the units column.

$28 = 5 \cdot 5 + \boxed{3} \Rightarrow$ Carry digit = 5

$13 + 5 = 18 = 5 \cdot 3 + \boxed{3} \Rightarrow$ Carry digit = 3

$21 + 3 = 24 = 5 \cdot 4 + \boxed{4} \Rightarrow$ Carry digit = 4

$18 + 4 = 22 = 5 \cdot 4 + \boxed{2} \Rightarrow$ Carry digit = $\boxed{4}$

Thus, the total is <u>**$42433_5$**</u>

Check: $4(5^4) + 2(5^3) + 4(5^2) + 3(5) + 3$

$= 4(625) + 2(125) + 4(25) + 3(5) + 3$

$= 2500 + 250 + 118 = 2868_{10}$

**Proof of the fact used to solve Question C in round 1**

Given the equations of two perpendicular lines,
the equations of the lines bisecting the opposite pairs of vertical angles can be found by
1) adding these equations, and
2) subtracting these equations.

<u>Proof</u>:
Perpendicular lines have negative reciprocal slopes.
Let the two lines have equations $\mathcal{L}_1 : ax + by = c_1$ (slope $-\dfrac{a}{b}$) and $\mathcal{L}_2 : bx - ay = c_2$ (slope $+\dfrac{b}{a}$).
Using the point-to-line distance formula $\left[\, d\big((h,k),\ Ax + by + C = 0\big) = \dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}} \,\right]$, we require that the arbitrary point $P$ be equidistant from the sides of the angle, i.e., $PQ = PR$.

$$d\big((x,y),\ ax + by - c_1 = 0\big) = d\big((x,y),\ bx - ay - c_2 = 0\big)$$

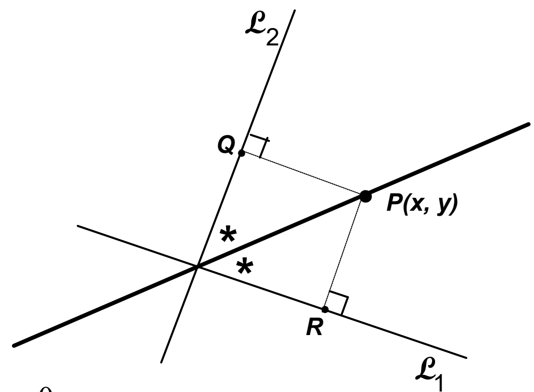
$$\Leftrightarrow \frac{|ax + by - c_1|}{\cancel{\sqrt{a^2 + b^2}}} = \frac{|bx - ay - c_2|}{\cancel{\sqrt{b^2 + (-a)^2}}}$$

$$\Leftrightarrow ax + by - c_1 = \pm\big(bx - ay - c_2\big)$$

$$\Leftrightarrow ax + by - c_1 = bx - ay - c_2 \quad \text{or} \quad ax + by - c_1 = -bx + ay + c_2$$

$$\Leftrightarrow$$

$$(a - b)x + (b + a)y - c_1 + c_2 = 0 \quad \text{or} \quad (a + b)x + (b - a)y - c_1 - c_2 = 0.$$




The first equation can be obtained by subtracting the corresponding terms in the original equations.

$$\boxed{\begin{array}{rr} & ax + by - c_1 = 0 \\ - & \underline{bx - ay - c_2 = 0} \end{array}}$$

The second equation is obtained by adding the corresponding terms in the original equations.
Q.E.D.
Latin for what Euclid often wrote at the conclusion of his proofs in "The Elements".
It stood for "quod erat demonstrandum".
Roughly translated - "which was to be demonstrated",
or, "We have proved to be true what we claimed was true."

**Hyperbola Equation Challenge <u>HINT</u>**:
To rotate a point $(x, y)$ about the origin, the rotation matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, so

$$(x_{new}, y_{new}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \boxed{\big(x(\cos\theta) - y(\sin\theta),\ x(\sin\theta) + y(\cos\theta)\big)}.$$

Details available on request. Send a request to olson.re@gmail.com.

# MASSACHUSETTS MATHEMATICS LEAGUE
# CONTEST 4 - JANUARY 2020 ANSWERS

**Round 1 Analytic Geometry: Anything**

A) $\left(64, 8\sqrt{2}\right)$  B) (12, 4)  C) $(1, -10)$

**Round 2 Algebra: Factoring**

A) $2(3x-2)(x-4)(x+4)$  B) 421  C) $-338$

**Round 3 Trig: Equations**

A) $\frac{3}{4}$  B) 225  C) $-\frac{1}{15}$

**Round 4 Algebra 2: Quadratic Equations**

A) $-1, 4$  B) 6 only  C) $4, -\frac{1}{72}$

**Round 5 Geometry: Similarity**

A) $\frac{32}{7}$  B) 153  C) $5\left(6+\sqrt{11}\right)$ or $30+5\sqrt{11}$

**Round 6 Algebra 1: Anything**

A) 14  B) 1 : 2  C) $\frac{5}{2} \le x < \frac{11}{2}$
(or $2.5 \le x < 5.5$ )

**Team Round**

A) $(-1, 4, 25, 24)$  D) 64, 100, 289

B) $-3, -1, 0, 1, 2$ (in any order)  E) $\frac{2\sqrt{7}}{3}$

C) $\frac{\sqrt{3}-1}{2}$  F) $42433_5$ (or 42433)