MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

ANSWERS

A) ________________________________

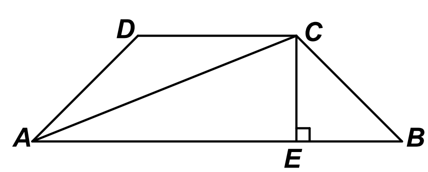
B) ________________________________

C) ________________________________

A) An isosceles (but not equilateral) triangle with integer dimensions has perimeter 12. Compute all possible values of the sine of a base angle.

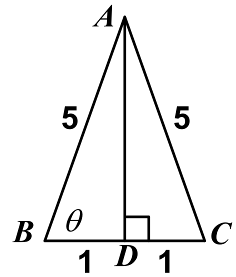
B) Compute $\tan\theta$, where θ is the largest angle in a triangle with sides of lengths 7, 8, and 13.

C) In isosceles trapezoid $ABCD$, $\cos(\angle CAE) = \dfrac{3}{4}$, $AC = 20$.
Compute the area of $ABCD$.



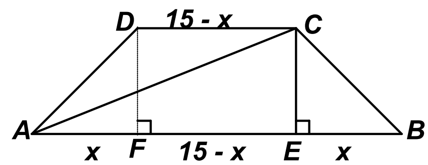
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019 SOLUTION KEY

Round 1



A) Impossible triangles: 1-1-10, 2-2-8, 3-3-6
Equilateral: 4-4-4
Only remaining possibility: 5-5-2
$AD=\sqrt{24} \Rightarrow \sin\theta = \underline{\mathbf{\frac{2\sqrt{6}}{5}}}$.

B) Since the largest angle must be opposite the longest side, using the law of cosines, we have
$13^2=7^2+8^2-2\cdot7\cdot8\cos\theta \Rightarrow \cos\theta=\frac{169-49-64}{-2\cdot7\cdot8}=\frac{\cancel{56}}{-2\cdot\cancel{7\cdot8}}=-\frac{1}{2} \Rightarrow \theta=120° \Rightarrow \tan\theta=\underline{\mathbf{-\sqrt{3}}}$.



C) $\cos(\angle CAE)=\frac{AE}{AC}=\frac{AE}{20}=\frac{3}{4} \Rightarrow AE=15$.
In ΔAEC , $CE^2=20^2-15^2=175 \Rightarrow CE=5\sqrt{7}$.
Drop a perpendicular from D to $\overline{AB}$.
Since $ABCD$ is an isosceles trapezoid, $AF=BE=x \Rightarrow FE=15-x$ and $AB=15+x$
The area of $ABCD$ is $\frac{1}{2}(5\sqrt{7})((15+x)+(15-x))=\frac{1}{2}(5\sqrt{7})(30)=\underline{\mathbf{75\sqrt{7}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019
ROUND 2 ARITHMETIC/NUMBER THEORY

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) In the puzzle below, each card hides a digit.
On the left, we have the sum of 996 and an unknown 2-digit natural number.
The 4-digit number on the right side of the equation is divisible by both 3 and 4.
Multiple sums of the rightmost two digits $(X+Y)$ are possible. Determine the only sums that occur more than once, for different X, Y-values.

996 + □□ = □□ **X** **Y**

B) Determine the *smallest* positive integer which can be written as the sum of two distinct primes in exactly three different ways.
Note:
3 + 11 and 11 + 3 are not considered different ways of writing 14.
7 + 7 is not allowed, since the primes used are not distinct.

C) (a, b, c) is a Pythagorean Triple. r_1, r_2 and r_3 are the respective remainders when a, b and c are divided by 4. Compute the minimum sum $a + b + c$ for which $r_1 + r_2 + r_3 = 6$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019 SOLUTION KEY

Round 2

A) Numbers which are multiples of both 3 and 4 are exactly those numbers divisible by 12. On the left side of the equation, $\square\square$ represents an integer between 10 and 99, inclusive. This means $996+10=1006 \le \square\square\boxed{X}\boxed{Y} \le 996+99=1085$, and the sum of the digits is $1+X+Y$. This sum must be a multiple of 3 to guarantee divisibility by 3. Divisibility by 4 requires that $\boxed{X}\boxed{Y}=10X+Y$ must be a multiple of 4. Thus, the rightmost two digits must be 08, 20, 32, 44, 56, 68, 80, or 92, which produce digit-sums of 8, 2, 5, 8, 11, 14, 8, 11. The recurring sums are **8** and **11**.

B) The smallest positive integer which can be written as the sum of two distinct primes is 5. All larger odd integers can be expressed as the sum of two primes in at most one way, so they do not need to be examined one at a time. Examining the even integers, we have
8 = 3 + 5 (unique)
10 = 3 + 7 (unique)
12 = 5 + 7 (unique)
14 = 3 + 11 (unique)
16 = 3 + 13 = 5 + 11
18 = 5 + 13 = 7 + 11
20 = 3 + 17 = 7 + 13
22 = 3 + 19 = 5 + 17
24 = 5 + 19 = 7 + 17 = 11 + 13 Bingo!

C) $(3,4,5) \Rightarrow r_1+r_2+r_3 = 3+0+1 = 4$.
$k(3,4,5)$ for k = 2, 3, 4, 5, 6, 7, 8, … $\Rightarrow r_1+r_2+r_3$ = 4, 4, 0, 4, 4, 4, 0, … .
$(5, 12, 13) \Rightarrow r_1+r_2+r_3 = 1+0+1 = 2$.
$k(5, 12, 13)$ for k = 2, 3, 4, …. $\Rightarrow r_1+r_2+r_3$ = 4, $\boxed{6}$, … .
Thus, 3(5, 12, 13) = (15, 36, 39) $\Rightarrow a+b+c =$ **90**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

ANSWERS

A) (______________ , ______________)

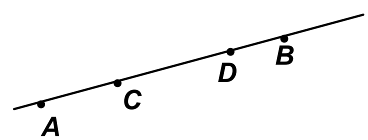
B) ________________________________

C) (____ , ____ , ____ , ____ , ____)

A) Consider the unit circle centered at the origin. Circle P is *externally* tangent to this unit circle at point B in the 4th quadrant, and to the x-axis and y-axis. Compute the coordinates of point P.

B) Given: $C(1,4)$ and $D(25,36)$

If $AC = \frac{1}{3}AB$ and $BD = \frac{1}{4}AB$, compute AB.

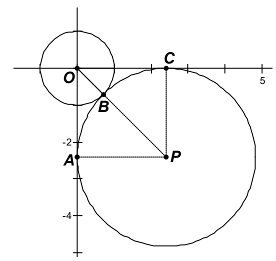


C) Circle O has its center at the intersection of the lines $2x - 3y + 6 = 0$ and $4x - 3y + 2 = 0$ and it passes through the point (1, 4). If its equation is put in the form $Ax^2 + By^2 + Cx + Dy + E = 0$, where A, B, C, D, and E are relatively prime integers and $A > 0$, compute (A, B, C, D, E).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019 SOLUTION KEY

Round 3

A) Since circle P is tangent to both axes, P must be equidistant from the axes. Let the coordinates of P be $(k,-k)$, for some $k>0$.
Then the lengths of the sides of right ΔAPO are k, k, and $k+1$.
Therefore, $k^2+k^2=(k+1)^2 \Rightarrow k^2-2k-1=0$.
Applying the quadratic formula, $k=\frac{2\pm\sqrt{4+4}}{2}=1+\sqrt{2},\ \cancel{1-\sqrt{2}}$.
$\underline{P\left(\mathbf{1+\sqrt{2},-1-\sqrt{2}}\right)}$.



B) $CD=\sqrt{(25-1)^2+(36-4)^2}$ or, recognizing the Pythagorean Triple $(24,32,40)=8(3,4,5)$, we have $CD=40$, but also $\frac{1}{3}AB+CD+\frac{1}{4}AB=AB \Leftrightarrow CD+\frac{7}{12}AB=AB$
$\Rightarrow \frac{5}{12}AB=40 \Rightarrow AB=8\cdot 12=\underline{\mathbf{96}}$.

C) $\begin{cases} 2x-3y+6=0 \\ 4x-3y+2=0 \end{cases} \Rightarrow 2x-4=0 \Rightarrow x=h=2, y=k=\frac{10}{3}$.
$(x-2)^2+\left(y-\frac{10}{3}\right)^2=r^2 \Rightarrow r^2=(1-2)^2+\left(4-\frac{10}{3}\right)^2=1+\frac{4}{9}=\frac{13}{9}$.
Expanding, $x^2-4x+4+y^2-\frac{20}{3}y+\frac{100}{9}=\frac{13}{9} \Leftrightarrow x^2+y^2-4x-\frac{20}{3}y+\frac{123}{9}=0$.
Multiplying through by 9, $9x^2+9y^2-36x-60y+123=0$.
Since these coefficients are not relatively prime, we must divide through by 3,
$\Rightarrow \underline{(\mathbf{3,3,-12,-20,41})}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019
ROUND 4 ALGEBRA 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) (____________ , ____________)

A) Compute n so that $x^{\frac{1}{3}} \cdot x^{\frac{1}{5}} \cdot x^{\frac{1}{n}} = \sqrt[10]{x^7}$, for $x > 1$.

B) Solve for x. $\log_4\left(\frac{1}{64}\right) - 3\log_{\frac{1}{1024}}(256) = \log_{32} x$

C) Let $f(x) = 2^x + 100\left(2^{-x}\right)$ and $g(x) = 4^x$

These functions intersect at the point $P(a,b)$, where a is in the form $\log_n m$.

Compute the ordered pair (a,b).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019 SOLUTION KEY

Round 4

A) $x^{\frac{1}{3}} \cdot x^{\frac{1}{5}} \cdot x^{\frac{1}{n}} = x^{\frac{5n+3n+15}{15n}} = x^{\frac{7}{10}}$.

Equating exponents, $\dfrac{8n+15}{15n} = \dfrac{7}{10}$.

Cross-multiplying, $80n+150 = 105n \Rightarrow 25n = 150 \Rightarrow n = \underline{\mathbf{6}}$.

B) $-3-3\left(-\dfrac{4}{5}\right) = \log_{32} x \Rightarrow \log_{32} x = -\dfrac{3}{5} \Rightarrow x = (32)^{-3/5} = \left(\dfrac{1}{32}\right)^{3/5} = \left(\dfrac{1}{2}\right)^{3} = \underline{\mathbf{\dfrac{1}{8}}}$.

C) $f(x) = g(x) \Leftrightarrow 2^x + 100\left(2^{-x}\right) = 4^x = \left(2^2\right)^x = 2^{2x} = \left(2^x\right)^2$

Thus, $\left(2^x\right)^2 - 2^x - \dfrac{100}{2^x} = 0 \Leftrightarrow \left(2^x\right)^3 - \left(2^x\right)^2 - 100 = 0$

By inspection, if $y = 2^x = 5$, we have 125 – 25 – 100 = 0. The quadratic factor, $y^2 + 4y + 20$, has only complex roots.

Since $2^x = 5 \Rightarrow 4^x = 25$, the point of intersection is $(a,b) = \underline{\mathbf{(\log_2 5,\ 25)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019
ROUND 5 ALGEBRA 1: RATIO, PROPORTION OR VARIATION

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) A, B, and C are points on the number line with coordinates $6-x$, $x+7$, and $11+x$, respectively. If B is between A and C and $\frac{AB}{BC}=\frac{1}{8}$, compute x.

B) In a scalene triangle, the measures of the two smallest interior angles are in a 3 : 2 ratio. The measures of the two largest interior angles are in a 5 : 2 ratio. Compute the degree-measure of the smallest *exterior* angle.

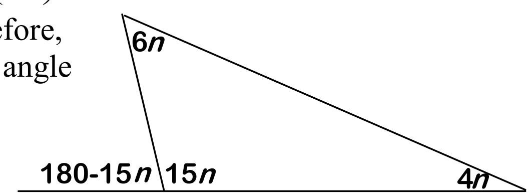
C) Suppose c varies jointly as the inverse of the square of a and the cube of the inverse of b. If $c=\frac{1}{8}$, when $(a,b)=(5,4)$, compute c, when $(a,b)=(4,5)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019 SOLUTION KEY

Round 5

A) $\dfrac{AC}{BC}=\dfrac{(x+7)-(6-x)}{(11+x)-(x+7)}=\dfrac{2x+1}{\cancel{4}^{1}}=\dfrac{1}{\cancel{8}^{2}}\Rightarrow 4x+2=1\Rightarrow x=\underline{\mathbf{-\frac{1}{4}}}$.

B) Let the 3 angle measures be $a<b<c$. Then $a:b=2:3$ and $c:b=5:2$.
If the first ratio is written as 4 : 6 and the second ratio as 15 : 6, then, in terms of n, we have $(a,b,c)=(4n,6n,15n)$. $25n=180\Rightarrow n=\dfrac{36}{5}\Rightarrow 15n=15\left(\dfrac{36}{5}\right)=108$.
Since the smallest exterior angle is adjacent to (and, therefore, the supplement of) the largest interior angle, the required angle has a degree-measure of **<u>72</u>**.



C) $c=\dfrac{k}{a^2b^3}\Rightarrow\dfrac{1}{8}=\dfrac{k}{5^24^3}$. Cross-multiplying, $k=\dfrac{5^24^3}{8}=25\cdot 8=200$
Substituting, $c=\dfrac{200}{4^2\cdot 5^3}=\dfrac{50}{4\cdot 5^3}=\dfrac{1}{2\cdot 5}=\underline{\mathbf{\frac{1}{10}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

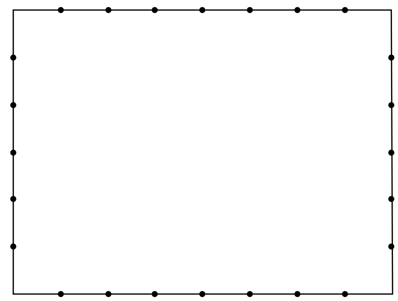
ANSWERS

A) ______________________________

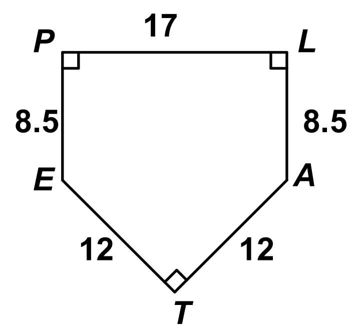
B) ________ , ________ , ________)

C) ______________________________

A) 24 points are located on the sides of a 6 x 8 rectangle in such a way that each side is divided into segments of unit length. P and Q are distinct points chosen from this set of 24 points. $\overline{PQ}$ crosses the interior of the 6 x 8 rectangle. How many distinct segments $\overline{PQ}$ have integer length.



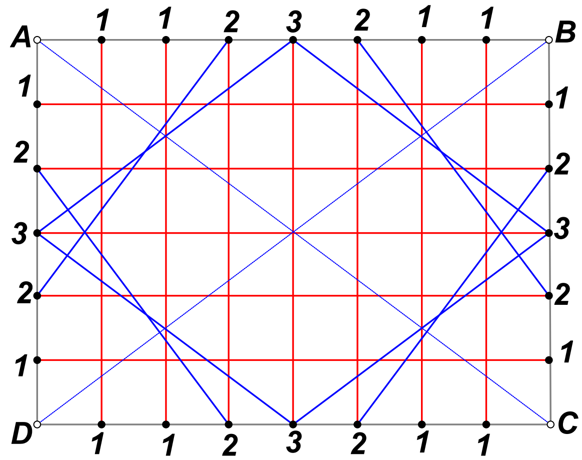
B) In baseball, according to the MLB rulebook, home plate has 3 right angles and dimensions shown at the right. Rules may be rules, but, as students of mathematics, we realize that this shape cannot exist. The actual perimeter of home plate (in inches), given the three right angles and the measurements of 17" and 8.5", is $A+B\sqrt{C}$. Compute the integer ordered triple (A,B,C).



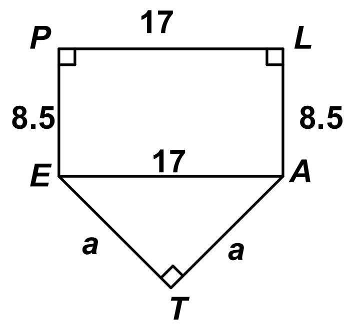
C) Regular polygon P has three times as many sides as regular polygon Q.
P has 976 more diagonals than Q and an interior angle of P is $k°$ larger than an interior angle of Q. Compute k.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019 SOLUTION KEY

Round 6



A) 5 horizontal/7 vertical/8 diagonal (3-4-5) $\Rightarrow$ **<u>20</u>** segments. The dotted lines are 10 units in length, but can't be counted since *A*, *B*, *C*, and *D* do not belong to the set of 24 points.
Alternately, each of the 24 points can be assigned the number of segments of integer length which have that point as an endpoint, but since each segment will be counted twice, we must divide the total by 2. $2(11)+2(9)=40,\ 40\div 2=\underline{\mathbf{20}}$.



B) $2a^2=17^2 \Rightarrow a^2=\frac{17^2}{2}=\frac{17^2\cdot 2}{4}\Rightarrow a=\frac{17}{2}\sqrt{2}$.
The perimeter of *PLATE* is $34+17\sqrt{2}\Rightarrow (A,B,C)=\underline{(\mathbf{34,17,2})}$.

C) $\#diag_P=\#diag_Q+976 \Leftrightarrow \frac{3n(3n-3)}{2}=\frac{n(n-3)}{2}+976$
$\Leftrightarrow 9n^2-9n=n^2-3n+2(976)$
$\Leftrightarrow 8n^2-6n-2(976)=0 \Leftrightarrow 4n^2-3n-976=0$.

Since the legit solution must be an integer, one factor must be of the form $n-\boxed{A}$ and the other must be of the form $4n+\boxed{B}$, where $AB=976$.
Factoring the constant term, we have $976=4(244)=16(61)$.
Clearly, $(n-16)(4n+61)$ gives the correct middle term, and Q has 16 sides.
Applying $\frac{180(n-2)}{n}$ and avoiding as much tedious arithmetic as possible, we have
$\frac{180\cdot 46}{48}-\frac{180\cdot 14}{16}=\frac{180}{48}(46-14\cdot 3)=\frac{15}{4}(4)=\underline{\mathbf{15}}$.
Even more tedious arithmetic can be avoided by looking at the calculations in terms of *exterior* angles of the regular polygon.
$\frac{360}{16}-\frac{360}{48}=360\left(\frac{1}{16}-\frac{1}{48}\right)=360\left(\frac{1}{24}\right)=\underline{\mathbf{15}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________ D) ____________________

B) (______ , ______ , ______) E) (__________ , __________)

C) ____________________ F) __________ : __________

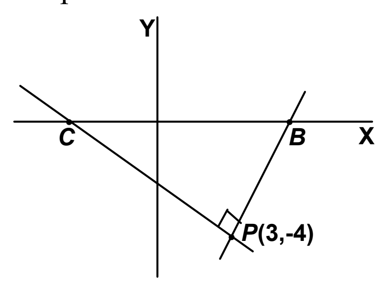
A) In ΔABC, $AC = 24$, $BC = 40$, and $AB = 56$. The bisector of $\angle C$ is extended through D on $\overline{AB}$ to point E, so that $DE = 9$. Compute AE.

B) We are all familiar with the primitive Pythagorean triples $(a,b,c)=\begin{cases}(3,\ 4,\ 5)\\(5,\ 12,\ 13)\\(7,\ 24,\ 25)\end{cases}$.

Observing the patterns between the terms in successive rows, we can predict additional rows. The next three rows would be (9, 40, 41), (11, 60, 61), and (13, 84, 85).

Here's a different pattern: $(a,b,c)=\begin{cases}(3,\ 4,\ 5)\\(5,\ 12,\ 13)\\(13,\ 84,\ 85)\end{cases}$, where a pattern is not so obvious.

Start with row 1: (3, 4, 5). Subsequent rows are *primitive* Pythagorean triples that start with largest number in the previous row and have $c-b=1$.
Compute the 4th row as an ordered triple (a, b, c), where $a < b < c$.



C) A line L_1 with slope m and x-intercept B passes through $P(3, -4)$.
A line L_2 passing through P perpendicular to L_1 has x-intercept C.
Find all possible values of m, if the area of ΔPBC is $\frac{52}{3}$.

D) Solve for x: $\log_2 1024 - 5\log_4 x + 3\log_8 x^2 = \dfrac{\log_3 x^2}{\log_3 16} + \log_{32} 1024$

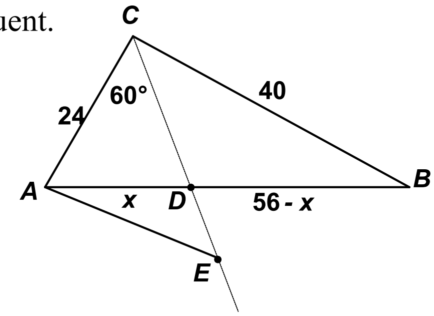
E) T varies jointly as p and the square of q, and inversely as the n^{th} power of r, where n is a positive integer. Let k denote the proportionality constant. $T = 10^6$, when $(p,q,r,n) = (40,25,2,3)$.
If p and q are both quadrupled (increased by a factor of 4), the value of T is unchanged when r is multiplied by the positive integer c.
For the maximum value of n, compute the ordered pair (k,c).

F) Given: decagon $V_1V_2...V_{10}$
Let C_1 be the number of distinct triangles formed by selecting any 3 vertices from $V_1, V_2, ..., V_{10}$.
Let C_2 be the number of distinct triangles all of whose sides are diagonals of $V_1V_2...V_{10}$.
Compute the ratio $C_1 : C_2$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019 SOLUTION KEY

Team Round

A) In ΔABC, $(AC, BC, AB) = (24, 40, 56) = 8(3,5,7)$, so ΔABC is similar to a triangle with side 3, 5 and 7, and corresponding angles of similar triangles are congruent.



Therefore, $\cos C = \dfrac{3^2+5^2-7^2}{2(3)(5)} = \dfrac{34-49}{30} = -\dfrac{1}{2} \Rightarrow m\angle C = 120°$

$\Rightarrow m\angle ACE = 60°$. Applying the angle bisector theorem,

$\dfrac{x}{24} = \dfrac{56-x}{40} \Leftrightarrow \dfrac{x}{3} = \dfrac{56-x}{5} \Leftrightarrow 5x = 168 - 3x \Rightarrow x = 21 \Rightarrow AD = 21$,

$BD = 35$. The length of the angle bisector CD is

$\sqrt{24(40)-21(35)}^{***} = \sqrt{3\cdot 5(8\cdot 8 - 7\cdot 7)} = \sqrt{15\cdot 15} = 15$. Since $DE = 9$, $CE = 24$

and ΔACE is isosceles with a vertex angle of 60°, i.e., ΔACE must be equilateral, and $AE =$ **<u>24</u>**.

*** The proof of the formula used here is included at the end of this solution key.
You might want to try to prove the formula on your own before looking at the proof at the end of the solution key. A proof utilizes the Angle Bisector Theorem and Stewart's Theorem.

B) Since there is no obvious pattern, our plan is to apply the Pythagorean Theorem directly.

$85^2 + n^2 = (n+1)^2 \Rightarrow 85^2 = 2n + 1 \Rightarrow n = \dfrac{85^2 - 1^2}{2} = \dfrac{(85+1)(85-1)}{2} = 86(42) = 3612$.

Thus, the required ordered triple is **<u>(85, 3612, 3613)</u>**.

<u>FYI</u>: If (a,b,c) is a primitive Pythagorean Triple, where $a<b<c$, then $c-b$ must be either a perfect square or twice a perfect square.

Common (and not so common) primitive PTs: $3-4-5$, $8-15-17$, $20-21-29$, $33-56-65$, ...

Suppose $(85, n, n+k)$ is a primitive PT. Then:

$85^2 + n^2 = (n+k)^2 \Rightarrow 85^2 = 2nk + k^2 \Rightarrow n = \dfrac{85^2 - k^2}{2k}$. To find all possible primitive Pythagorean Triples with 85 as the short leg, we have to check $k = 1, 9, 25, 49$, and 81.

We have already found the triple for $k = 1$. $k = 9$, 49, 81 fail to give an integer value for n.

Checking $k = 25$, we have

$$n = \frac{85^2 - 25^2}{2\cdot 25} = \frac{(85+25)(85-25)}{50} = \frac{110\cdot 6\not{0}}{5\not{0}} = 22\cdot 6 = 132$$

Thus, we have the primitive PT $(85, 132, 157)$.

Other k-values might produce PTs, but they will not be primitive.

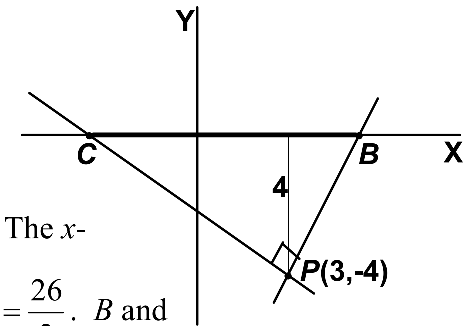
Using a calculator, you can verify that

$k = 5 \Rightarrow (85, 720, 725)$, but this triple has a common factor of 5.

$k = 17 \Rightarrow (85, 204, 221)$, but this triple has a common factor of 17.

Thus, neither of these are primitive, and there are exactly two primitive PTs with $a = 85$.

Team Round - continued



C) The equation of L_1 is $(y+4)=m(x-3) \Leftrightarrow mx-y=3m+4$

Letting $y=0$, the x-intercept is $\dfrac{3m+4}{m}$

The equation of L_2 is $(y+4)=-\dfrac{1}{m}(x-3) \Leftrightarrow x+my=3-4m$. The x-intercept is 3 – 4m. The area of ΔPBC is $\dfrac{1}{2}\cdot 4\cdot BC=\dfrac{52}{3} \Rightarrow BC=\dfrac{26}{3}$. B and C lie on the x-axis, *but we don't know whether B lies to the right or left of C*. Thus, BC must be represented as the <u>absolute value</u> of the difference of their x-coordinates.

$$\left|\frac{3m+4}{m}-(3-4m)\right|=\frac{26}{3} \Leftrightarrow \left|\frac{3m+4-3m+4m^2}{m}\right|=\frac{26}{3}$$

$$\Leftrightarrow \frac{2\left|m^2+1\right|}{|m|}=\frac{13}{3} \Leftrightarrow 6\left|m^2+1\right|=13|m| \Leftrightarrow 6m^2 \pm 13m+6=0.$$

Factoring, $(2m-3)(3m-2)=0$ or $(2m+3)(3m+2)=0$. Therefore, $m = \underline{\pm\dfrac{3}{2}, \pm\dfrac{2}{3}}$.

D) Since the rightmost and leftmost terms are just constants, and the fractional term can be simplified to $\log_{16} x^2$, using base conversion, we have $10-5\log_4 x+6\log_8 x=2\log_{16} x+2$.

Now we convert each log expression to $\log_4 x$.

$$\log_8 x=y \Leftrightarrow 8^y=x \Leftrightarrow \left(4^{\frac{3}{2}}\right)^y=4^{\left(\frac{3}{2}y\right)} \Leftrightarrow \frac{3}{2}y=\log_4 x \Leftrightarrow \boxed{y=\log_8 x=\frac{2}{3}\log_4 x}$$

$$\log_{16} x=y \Leftrightarrow 16^y=x \Leftrightarrow \left(4^2\right)^y=4^{(2y)} \Leftrightarrow 2y=\log_4 x \Leftrightarrow \boxed{y=\log_{16} x=\frac{1}{2}\log_4 x}$$

$$\Rightarrow 8-5\log_4 x+4\log_4 x-\log_4 x=0 \Leftrightarrow \log_4 x=4 \Leftrightarrow x=4^4=\underline{\mathbf{256}}.$$

Team Round - continued

E) (1) $T = k \cdot \dfrac{pq^2}{r^n}$

$(T,p,q,r,n) = (10^6, 40, 25, 2, 3) \Rightarrow k = \dfrac{10^6}{\dfrac{40(25)^2}{2^3}} = \dfrac{10^6}{5^5} = 2^6 \cdot 5 = 320$.

(2) $T = k\dfrac{(4p)(4q)^2}{(cr)^n}$. Equating the two expressions for T, $k \cdot \dfrac{pq^2}{r^n} = k\dfrac{(4p)(4q)^2}{(cr)^n}$,

we have $1 = \dfrac{64}{c^n} \Rightarrow c^n = 64 \Rightarrow (c,n) = (2,6), (4,3), (8,2), (64,1)$.

Thus, for the *maximum* value of n, $(k,c) = \underline{\mathbf{(320,2)}}$.

F) There are $\dfrac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} = 120$ possible triangles.

Let the vertices $V_1, V_2, ..., V_{10}$ be denoted *ABCDEFGHIJ.*

Any segment which does not connect two consecutive vertices is a diagonal.

Starting with *A*, we have:

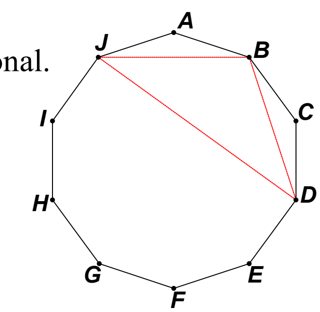
ACE, ACF, ACG, ACH, ACI

ADF, ADG, ADH, ADI

AEG, AEH, AEI

AFH, AFI

AGI



A total of 15 triangles, all of whose sides are diagonals.

There are 15 more triangles for each of these starting trios: *BDF*, *CEG*, *DFH*, *EGI*, and *FHJ*.

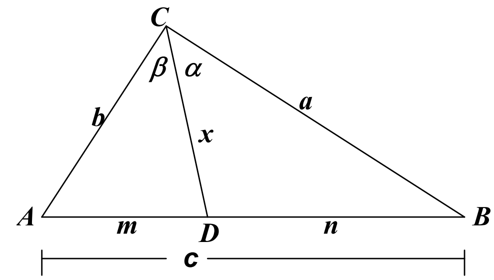
For example, the *BDF* sequence ends with *BJD* .

Thus, there are $6\cdot 15 = 90$ triangles formed strictly by diagonals.

The required ratio is 120 : 90 = **<u>4 : 3</u>**.

Length of Angle bisector Theorem $\boxed{x=\sqrt{ab-mn}}$

Prerequisite Theorems:



1) Angle bisector Theorem: If $\alpha=\beta$, then $\frac{m}{b}=\frac{n}{a}$.

2) Stewart's Theorem: $a^2m+b^2n=x^2c+cmn$

1) is a well known result proved by *similar triangles* - A proof might start out by drawing a line through *B* parallel to $\overline{AC}$ which intersects ray $\overrightarrow{CD}$ in point *E*.

For 2), *D* can be ANY point on $\overline{AB}$. A segment connecting a vertex of a triangle with ANY point on the opposite side is called a ***cevian***. The word was coined in honor of Italian mathematician Giovanni Ceva (1647 - 1734) who is credited with the first proof of a theorem named after him*. Some cevians are angle bisectors, some are medians, some are altitudes, and some are none of the above. Stewart's theorem specifies a relationship between the length of the cevian and the lengths of two sides of the triangle and the lengths of the segments on the third side. A proof might start out using the Law of Cosines on triangles *ACD* and *BCD*, and writing expressions for a^2 and b^2 using the supplementary angles at *D*.

Now on to the proof of the main attraction.

1) $\Rightarrow\ m=\frac{nb}{a}$ and $n=\frac{ma}{b}$

Since $c=m+n$, according to Stewart's Theorem, we have

$$a^2m+b^2n=x^2(m+n)+mn(m+n)\Rightarrow x^2=\frac{a^2m+b^2n}{m+n}-mn$$

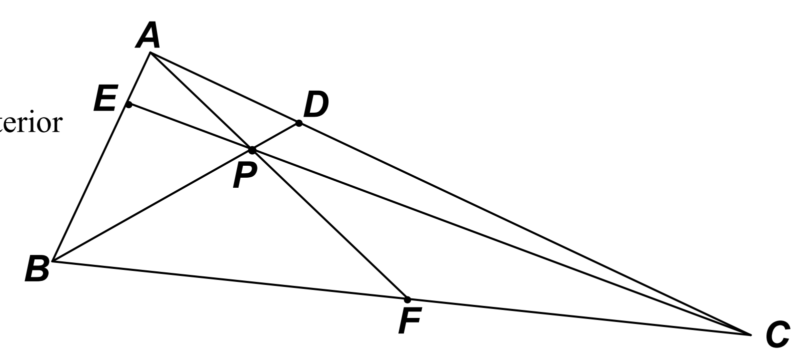
Substituting for *m* and *n* in the denominator, we have

$$x^2=\frac{a^2m+b^2n}{\frac{bn}{a}+\frac{am}{b}}-mn=\frac{a^2m+b^2n}{\frac{b^2n+a^2m}{ab}}-mn=ab-mn$$

There you have it - $\boxed{x=\sqrt{\mathbf{ab-mn}}}$.

Try verifying in *two different ways* that the angle bisector of the right angle in a 3-4-5 triangle has length $\frac{12\sqrt{2}}{7}$.

* Ceva's Theorem: Consider three cevians which intersect at a common point in the interior of a triangle. Then: $\boxed{\frac{AE}{BE}\cdot\frac{BF}{CF}\cdot\frac{CD}{AD}=1}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2019 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) $\frac{2\sqrt{6}}{5}$ B) $-\sqrt{3}$ C) $75\sqrt{7}$

Round 2 Arithmetic/Elementary Number Theory

A) 8, 11 B) 24
[5+19, 7+17, 11 + 13] C) 90

Round 3 Coordinate Geometry of Lines and Circles

A) $\left(1+\sqrt{2},-1-\sqrt{2}\right)$ B) 96 C) $\left(3,3,-12,-20,41\right)$

Round 4 Algebra 2: Log and Exponential Functions

A) 6 B) $\frac{1}{8}$ C) $\left(\log_2 5,\ 25\right)$

Round 5 Algebra 1: Ratio, Proportion or Variation

A) $-\frac{1}{4}$ B) 72 C) $\frac{1}{10}$

Round 6 Plane Geometry: Polygons (no areas)

A) 20 B) $\left(34,17,2\right)$ C) 15

Team Round

A) 24 D) 256

B) (85, 3612, 3613) E) (320, 2)

C) $\pm\frac{3}{2},\pm\frac{2}{3}$ F) 4 : 3