MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019
ROUND 1 COMPLEX NUMBERS (No Trig)

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) If $\frac{7i^7 - 13i^{13} + 18i^{19}}{\left(4+i\sqrt{3}\right)\left(4-i\sqrt{3}\right)} = A + Bi$, compute $A^2 + B^2$.

B) $\left(\sqrt{3}+i\right)^4 = x + yi$. Compute $\frac{y^2}{x^3}$.

C) D_1 and D_2 are complex roots of $8x^3 - 1 = 0$. Compute $\left(D_1^{\,2} + D_1^{\,4} + D_1^{\,8}\right) + \left(D_2^{\,2} + D_2^{\,4} + D_2^{\,8}\right)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019 SOLUTION KEY

Round 1

A) $$\frac{7i^7-13i^{13}+18i^{19}}{\left(4+i\sqrt{3}\right)\left(4-i\sqrt{3}\right)}=\frac{-7i-13i-18i}{16+3}=\frac{-38i}{19}=0-2i\Rightarrow A^2+B^2=\underline{\mathbf{4}}\,.$$

B) $$\left(\sqrt{3}+i\right)^4=\left(\left(\sqrt{3}+i\right)^2\right)^2=\left(2+2\sqrt{3}i\right)^2=4+8\sqrt{3}i-12=-8+8\sqrt{3}i$$

$$\frac{y^2}{x^3}=\frac{\left(8\sqrt{3}\right)^2}{(-8)^3}=\frac{8^2\cdot 3}{-(8)^3}=\underline{-\frac{\mathbf{3}}{\mathbf{8}}}\ .$$

C) $$8x^3-1=0\Leftrightarrow(2x-1)\left(4x^2+2x+1\right)=0\Rightarrow \text{(1) } D_1+D_2=-\frac{1}{2},\ \text{(2) } D_1D_2=\frac{1}{4}.$$

Regrouping, $$\left(D_1^{\,2}+D_1^{\,4}+D_1^{\,8}\right)+\left(D_2^{\,2}+D_2^{\,4}+D_2^{\,8}\right)=\boxed{\underline{\left(D_1^{\,2}+D_2^{\,2}\right)}+\underline{\underline{\left(D_1^{\,4}+D_2^{\,4}\right)}}+\underline{\underline{\underline{\left(D_1^{\,8}+D_2^{\,8}\right)}}}}\,.$$

Each of these sums can be determined from the sum and product of the roots of the given equation, without *bothering to determining the actual values of the roots*.

$$\left(D_1+D_2\right)^2=\underline{D_1^{\,2}+D_2^{\,2}}+2D_1D_2\Leftrightarrow\frac{1}{4}=\left(D_1^{\,2}+D_2^{\,2}\right)+\frac{1}{2}\Rightarrow D_1^{\,2}+D_2^{\,2}=-\frac{1}{4}.$$

$$\left(D_1^{\,2}+D_2^{\,2}\right)^2=\underline{\underline{D_1^{\,4}+D_2^{\,4}}}+2\left(D_1D_2\right)^2\Leftrightarrow\frac{1}{16}=\left(D_1^{\,4}+D_2^{\,4}\right)+\frac{1}{8}\Rightarrow D_1^{\,4}+D_2^{\,4}=-\frac{1}{16}.$$

$$\left(D_1^{\,4}+D_2^{\,4}\right)^2=\underline{\underline{\underline{D_1^{\,8}+D_2^{\,8}}}}+2\left(D_1D_2\right)^4\Leftrightarrow\frac{1}{256}=\left(D_1^{\,8}+D_2^{\,8}\right)+\frac{1}{128}\Rightarrow D_1^{\,8}+D_2^{\,8}=-\frac{1}{256}.$$

$$-\frac{1}{4}-\frac{1}{16}-\frac{1}{256}=-\frac{64+16+1}{256}=\underline{-\frac{\mathbf{81}}{\mathbf{256}}}\,.$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019
ROUND 2 ALGEBRA 1: ANYTHING

ANSWERS

A) (_____________ , _____________)

B) ______________________________

C) ______________________________

A) In a collection of nuts and bolts, the ratio of nuts to bolts is 7 : 6. If a single washer is added to the collection, there are 4200 pieces in the collection. If I had k more bolts, I would have exactly $j \cdot k$ pairs of nuts and bolts. Compute the ordered pair of positive integers (j,k).

B) What is the remainder when 24688642978 is divided by 18?

C) A, B and C are distinct integers selected from 1 to 9, inclusive.
The expression $A \cdot B - C$ produces a multiple of 10 for k ordered triples (A,B,C).
Compute k.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019 SOLUTION KEY

Round 2

A) $7x+6x+1=4200\Rightarrow x=\dfrac{4199}{13}=323$.

There is no need to compute the number of nuts and bolts.
I need x more bolts to give me $7x$ of each. Thus, $x=k=323$ and $(j,k)=\underline{(\mathbf{7,323})}$.

Check: Originally, there were 7(323) = 2261 nuts and 6(323) = 1938 bolts,
a total of 4199 pieces.

B) Can brute force long division be avoided? Of course; here's how.

$$\frac{N}{2}=\frac{24688642978}{2}=Q=12344321489$$

The sum of the digits of Q is 41, so Q is not divisible by 9; in fact, it leaves a remainder of 5.

Thus, $\dfrac{N}{2}=Q=9x+5\Rightarrow N=18x+10$, and the original number is 10 more than a multiple of 18.

Therefore, the required remainder is **10**.

Sure beats $\begin{array}{r} 1\ 3\ 71\ 5\ \ 912\ \ 7\ 6 \\ 18\overline{)\,24^{6}6^{12}8^{2}8^{10}6^{16}4^{2}2^{4}9^{13}7^{11}8} \end{array}$ $118-6(18)=118-108\Rightarrow r=\underline{\mathbf{10}}$.

C) If A = 1, no multiples of 10 can be generated.

1 2 3 4 5 6 7 8 9

Plan: Let A take on values from 2 through 8. For each A-value, moving through larger values in the above list picking B-values avoids duplication. Compute the product and ask "Is there is a distinct C-value which produces a difference whose units digit is zero?".

Following are the results for $B > A$.

$2\cdot3-6$ $2\cdot4-8$ $2\cdot7-4$ $2\cdot8-6$ $2\cdot9-8$

$3\cdot4-2$ $3\cdot6-8$ $3\cdot7-1$ $3\cdot8-4$ $3\cdot9-7$

$4\cdot7-8$ $4\cdot8-2$ $4\cdot9-6$

$6\cdot7-2$ $6\cdot9-4$

$7\cdot8-6$ $7\cdot9-3$

$8\cdot9-2$

If $A > B$, we get different ordered triples which can be put on a 1-to-1 correspondence with those listed above. Thus, the total number of ordered triples is **36**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

ANSWERS

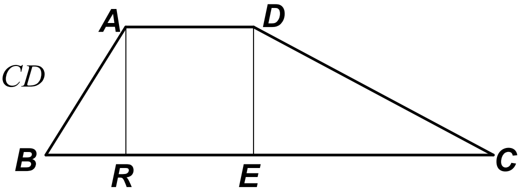
A) _______________ : _______________

B) (______________ , ______________)

C) __________________________________

A) Given rectangle *ABCD* with $A(-20.1, 20.19)$, $C(2.72, 3.14)$, and point *E*, which divides $\overline{AD}$ into a 3 : 2 ratio. Compute the ratio of the sum of the areas of ΔABE and ΔDCE to the area of rectangle *ABCD*.

B) Given: *DARE* is a square of area 81,
ABCD is a trapezoid, and $BR : CE = 3 : 10$.
Compute the ordered pair (BR, CE) so that the area of *ABCD* is 315 square units.

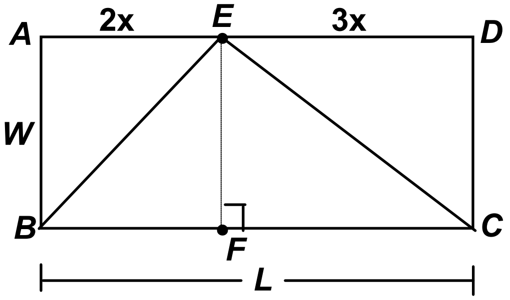


C) Point *P* is in the interior of square *ABCD*, whose area is greater than 2.
$PA = PB = \sqrt{3}$ and $PC = PD = 1$
Compute the area of *ABCD*.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019 SOLUTION KEY

Round 3

A) The coordinates and the ratio are irrelevant.
If the dimensions of the rectangle are L x W, the area of ΔBEC is $\frac{1}{2}LW$, and, by subtraction, the required sum is also $\frac{1}{2}LW$. Thus, the required ratio is **1 : 2**.

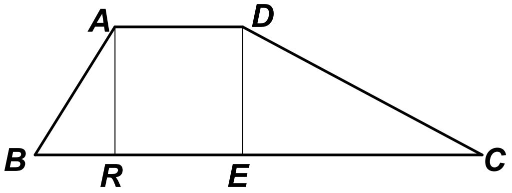


B) $AD = 9$. Let $(BR, CE) = (3a, 10a)$. Then:

$$\frac{1}{2}\cdot 9\cdot(9+(9+13a)) = 315$$

$$\Rightarrow 18+13a = \frac{630}{9} = 70 \Rightarrow 13a = 52$$

$$\Rightarrow a = 4 \Rightarrow (BR, CE) = \underline{(\mathbf{12,40})}.$$



C) Consider the diagram at the right of square $ABCD$, and an appropriately positioned point P.

$$AE = \sqrt{3-h^2},\ DE = \sqrt{1-h^2}$$

$$2h = AB = AD = \sqrt{3-h^2} + \underline{\sqrt{1-h^2}}$$

Transposing the underlined radical, and squaring both sides,

$$4h^2 - 4h\sqrt{1-h^2} + (1-h^2) = 3-h^2$$

$$4h^2 - 2 = 4h\sqrt{1-h^2} \Leftrightarrow \boxed{2h^2 - 1 = 2h\sqrt{1-h^2}}$$

Squaring both sides again,

$$4h^4 - 4h^2 + 1 = 4h^2(1-h^2) = 4h^2 - 4h^4$$

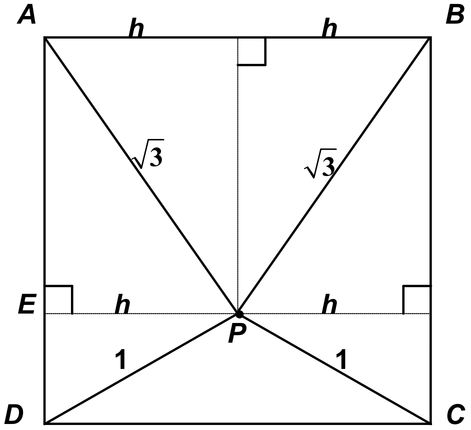
$$\Rightarrow 8h^4 - 8h^2 + 1 = 0.$$

Since this is a quadratic equation (in h^2), we have

$$h^2 = \frac{8 \pm \sqrt{64-32}}{16} = \frac{2\pm\sqrt{2}}{4} \Rightarrow h = \frac{\sqrt{2\pm\sqrt{2}}}{2}$$

$$\Rightarrow AB = \sqrt{2\pm\sqrt{2}} \Rightarrow AB^2 = \underline{\mathbf{2+\sqrt{2}}}.$$

[AB^2 can't be $2-\sqrt{2}$, since this value is less than 2.]



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019
ROUND 4 ALGEBRA 1: FACTORING AND ITS APPLICATIONS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) For how many ordered pairs (x, y) is the condition $x^2 y = xy^2$ satisfied, if $x, y \varepsilon \{0, 1, 2, \ldots, 100\}$?

B) Compute the <u>maximum</u> 2-digit integer k for which $4k$ has exactly 16 factors.

C) For how many integer values of x is $x^2\left(x^4+16\right) \le 17x^4$?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019 SOLUTION KEY

Round 4

A) $x^2y = xy^2 \Leftrightarrow xy(x-y)$

$xy(x-y)=0 \Leftrightarrow x = 0, y = 0$ or $x = y$.

There are 101 ordered pairs where $x = y$.

There are 200 ordered pairs, where exactly one coordinate is 0, namely (0, *) and (*, 0), where * denotes a nonzero integer. Thus, there are **<u>301</u>** such ordered pairs.

B) We can count the number of factors any integer has without actually listing them. By way of example, $24 = 2^3 3^1$ has $(3+1)(1+1) = 8$ factors, since factors of 24 can only have prime factors of 2 and 3, and the exponent of 2 can only be 0,1,2, or 3, and the exponent of 3 can only be 0 or 1. In general, the exponents can range from 0 to the maximum number of times the prime factor occurs in the prime factorization of *N*.

4·99	$2^2 \cdot 3^2 \cdot 11$	18
4·98	$2^3 \cdot 7^2$	12
4·97	$2^2 97^1$	6
4·96	$2^7 3^1$	16

$\Rightarrow k =$ **<u>96</u>**.

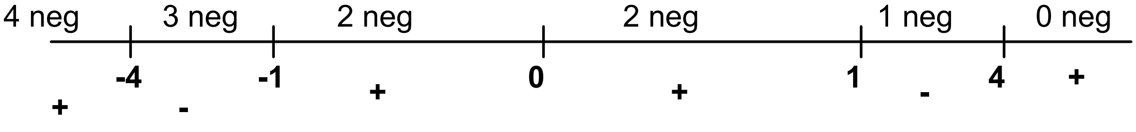
C) $x^2\left(x^4+16\right) \le 17x^4 \Leftrightarrow x^2\left(x^4-17x^2+4^2\right) \le 0$

$\Leftrightarrow x^2\left(\left(x^2-4\right)^2-9x^2\right) \le 0$ As the difference of perfect squares, this is equivalent to

$x^2\left(\left(x^2+3x-4\right)\left(x^2-3x-4\right)\right) \le 0$

$x^2(x+4)(x-1)(x-4)(x+1) \le 0$

Thus, the critical points are at $0, \pm 1, \pm 4$ and the sign of the product is given by



There is no sign change at $x = 0$, since the factor x occurs twice $\left(x^2 = (x-0)^2\right)$.

Thus, the solution set is $-4 \le x \le -1$ or $x = 0$ or $1 \le x \le 4$, which contains **<u>9</u>** integer values.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES

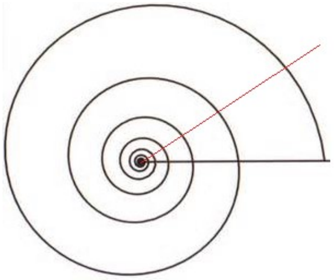
ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Coterminal angles share the same initial and terminal sides, but differ in the number of complete revolutions the initial side makes before it stops (on the terminal side). Therefore, angles can measure over 360°. Angles can have positive and negative measures, since revolutions can be either clockwise (as shown in the diagram at the right) or counterclockwise.



How many angle measures between -1,000° and 10,000° are coterminal with 30°? Include 30° in the count.

B) Evaluate $\sin\left(-1110^\circ\right)+\cos\left(1680^\circ\right)+\tan^2\left(150^\circ\left(2A-1\right)\right)$, where A denotes the 6th prime.

C) $N=-2\sin\frac{5\pi}{4}\cdot\left(\tan\left(-\frac{2\pi}{3}\right)-\cos\left(2k-1\right)\pi\right)$, where k is an integer.

For some integer a, $a<N<a+1$. Compute a.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019 SOLUTION KEY

Round 5

A) $-1000 < 30 + 360k < 10000$

$\Leftrightarrow \frac{-1030}{360} < k < \frac{9970}{360}$

$\Leftrightarrow -\frac{103}{36} < k < \frac{997}{36} \Leftrightarrow -2^{-} < k < 27^{+}$.

Therefore, $k = -2, -1, 0, 1, 2, 3, 4, 5, ..., 27$, a total of $\underline{\mathbf{30}}$ values.

B) $-1110° + 3(360°) = 1080° - 1110° = -30° \Rightarrow \sin(-30°) = -\sin(30°) = -\frac{1}{2}$

$1680° = 4(360°) + 240° \Rightarrow \cos(240°) = -\cos(60°) = -\frac{1}{2}$

$2, 3, 5, 7, 11,\ A = \underline{13},\ 150°(25) = 3750° - 10(360°) \Rightarrow 150°,\ \tan^2(150°) = \left(-\frac{\sqrt{3}}{3}\right)^2 = \frac{1}{3} \Rightarrow \underline{-\frac{\mathbf{2}}{\mathbf{3}}}$.

C) For integer values of *k*, the expression $(2k-1)\pi$ generates odd multiples of *π*, where the cosine always has a value of –1.

$\frac{5\pi}{4}$ is the $\frac{\pi}{4}$ family (in quadrant 3 where the sine is negative)

$-\frac{2\pi}{3}$ is the $\frac{\pi}{3}$ family (also in quadrant 3 – turning clockwise – where the tangent is positive)

Thus, $N = -2\left(-\frac{\sqrt{2}}{2}\right)\left(\sqrt{3}+1\right) = \sqrt{6} + \sqrt{2} \Rightarrow a = \underline{\mathbf{3}}$.

Here is a more detailed argument:

To approximate $\sqrt{6} + \sqrt{2}$, square the expression, getting $8 + 4\sqrt{3}$.

$1 < 3 < 4 \Rightarrow 1 < \sqrt{3} < 2 \Rightarrow 4 < 4\sqrt{3} < 8 \Rightarrow 12 < 8 + 4\sqrt{3} < 16$.

Thus, the square of the desired value lies between 12 and 16 and must be "3.something" $\Rightarrow a = \underline{\mathbf{3}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019
ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS

ANSWERS

A) ______________________________

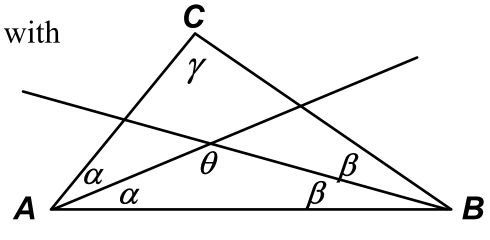
B) ______________________________

C) ______________________________

A) In parallelogram *ABCD*, $m\angle DAB = 3(3x-1)$ and $m\angle DCB = (x+1)(x+2)$.
Compute <u>both</u> possible measures of $\angle ADC$.

B) The angles in scalene triangle *PET* have measures in a ratio of 7 : 9 : 14.
The bisector of the smallest angle and the trisectors of the largest angle divide PET into 6 disjoint, i.e., non-overlapping, regions, two of which are quadrilaterals. List from smallest to largest the degree-measures of the four angles of the quadrilateral containing the largest angle. *Do <u>not</u> include the degree symbols in your answer.*

C) In acute scalene triangle ABC, α, β, γ, and θ are angles with integral degree measures,
$m\angle CAB : m\angle CBA = 6:5$, $\gamma < 2\alpha$, $\theta = m°$, $\gamma = n°$.
If $\frac{m}{n} = \frac{m'}{n'}$, a reduced fraction, compute <u>all</u> possible ordered pairs (m', n').



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019 SOLUTION KEY

Round 6

A) Since the given angles are opposite angles in a parallelogram, and opposite angles always have equal measures, we have

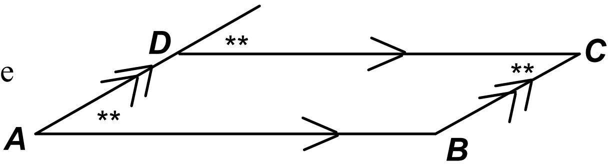
$(x+1)(x+2)=3(3x-1) \Leftrightarrow x^2+3x+2=9x-3 \Leftrightarrow x^2-6x+5=0$

$\Leftrightarrow (x-1)(x-5)=0 \Rightarrow x=1,\ 5$.

The required angle is one of the other pair of opposite angles. Each of these angles is supplementary to an angle in the first pair.

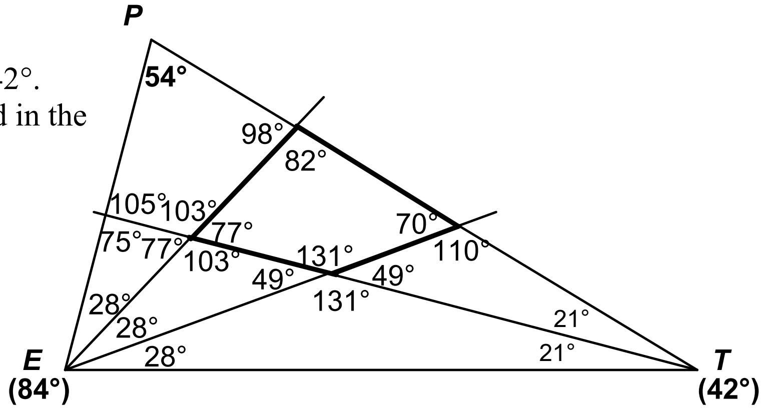
$x=1 \Rightarrow m\angle A=3(3-1)=6° \Rightarrow m\angle ADC=\underline{\mathbf{174°}}$.

$x=5 \Rightarrow m\angle A=3(15-1)=42° \Rightarrow m\angle ADC=\underline{\mathbf{138°}}$.



B) $7n+9n+14n=30n=180 \Rightarrow n=6$
$\Rightarrow$ Biggest angle: 84°, Smallest angle: 42°.
The angle measures in PET are indicated in the diagram at the right.

$\Rightarrow$ **70, 77, 82, 131**.



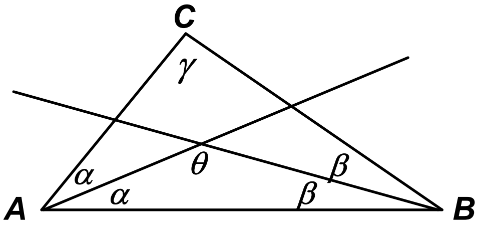
C) Let $(\alpha,\beta)=(3x,2.5x)$.

Angle $A=2\alpha$ is acute only if $x<15$.

β has integer measure only if x is even.

$\gamma<2\alpha \Rightarrow 180-11x<6x \Rightarrow 17x>180 \Rightarrow x\geq 11$.

Therefore, x must be either 12 or 14.



$$x=12 \Rightarrow (\alpha,\beta,\gamma)=(36°,30°,48°),\ \theta=180-(36+30)=114 \Rightarrow \frac{m}{n}=\frac{114}{48}=\frac{19}{8} \Rightarrow (m',n')=\underline{(\mathbf{19,8})}.$$

$$x=14 \Rightarrow (\alpha,\beta,\gamma)=(42°,35°,26°),\ \theta=180-(42+35)=103 \Rightarrow \frac{m}{n}=\frac{103}{26} \Rightarrow (m',n')=\underline{(\mathbf{103,26})}.$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) (___________ , ___________)

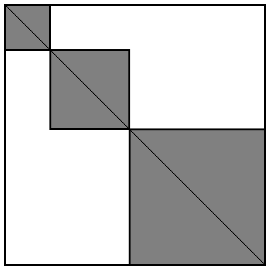
B) ____________________________ E) ____________________________

C) ____________ : ____________ F) ____

A) Let $P_1(a,b)$ be a point in the complex plane corresponding to $a+bi=3+4i$.

Let $P_n(a,b)$ be a point in the complex plane corresponding to $P_{n-1}(a,b)\cdot\dfrac{\sqrt{3}+i}{2}$, for $n>1$.

This sequence of points determines a polygon whose area is *K* units^2. Compute *K*.



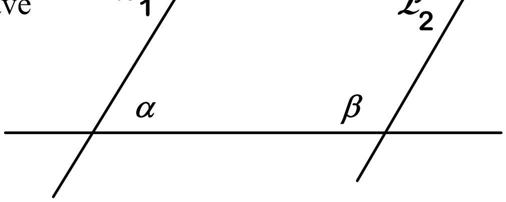
B) The diagram at the right shows three squares whose opposite vertices lie along the diagonal of a larger square.
Compute the *smallest* possible integer length of a side of the smallest square, if the ratio of the area of the unshaded region to the area of the shaded region is greater than 1.9, given that the sides of the shaded squares are x, $x+2$, and $x+4$.

C) In regular pentagon *MAPLE*, perpendiculars are drawn to diagonal $\overline{AE}$ from vertices *M*, *P*, and *L*, intersecting the diagonal in points *G*, *S*, and *R*, respectively. Compute the ratio of the area of triangle *MAG* to the area of rectangle *PLRS*.

D) The largest negative integer value and the smallest positive integer value of *n* for which the digit in the tenth's place in the expansion of the product $10^{\left|2n^2+n-28\right|}\left(0.\overline{2013}+0.\overline{11011}\right)$ is 0 are *L* and *S*, respectively.
Compute the ordered pair (*L*, *S*).

E) Solve for *x* over $0° < x < 360°$: $\sin^2\left(165°-x\right)+\cos^2\left(2x-195°\right)=1$

F) Let N and N' be two-digit base ten integers whose digits have been reversed, i.e., if $N=10x+y$, then $N'=10y+x$.
Let $m\angle\alpha=N$ and $m\angle\beta=kN'$, where *k* is an integer and $1\le k\le 9$. Compute all ordered triples (k,x,y) for which $\mathscr{L}_1$ and $\mathscr{L}_2$ are parallel.

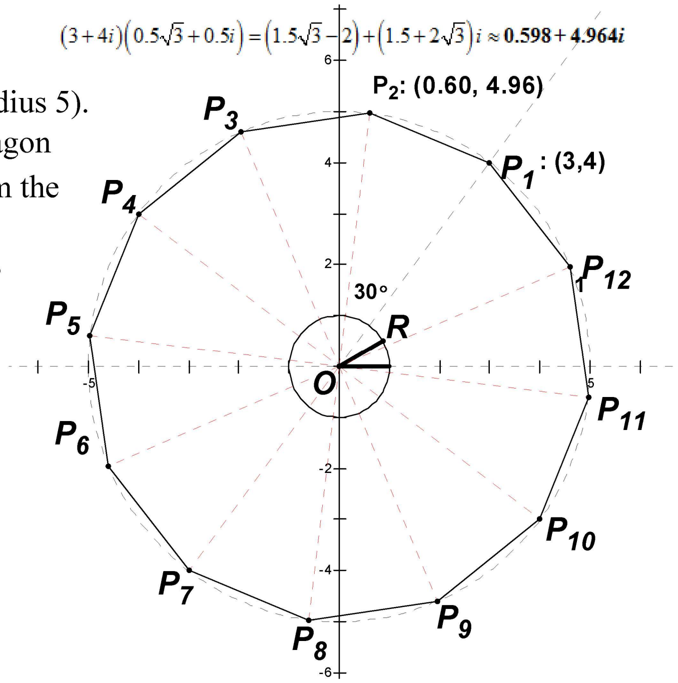


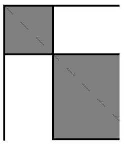
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019 SOLUTION KEY

Team Round

A) $\dfrac{\sqrt{3}+i}{2}$ corresponds to a point R in the complex plane on a unit circle centered at the origin (or pole). Point $R\left(\dfrac{\sqrt{3}}{2},\dfrac{1}{2}\right)$ determines a 30° angle in standard position. (See the diagram below.)

Multiplying by $\dfrac{\sqrt{3}+i}{2}$ "rotates" a given point 30° around the origin (in this case around a circle of radius 5). Thus, the P_n sequence determines a regular dodecagon (i.e., a regular 12-sided polygon) with radius 5 from the center point (0, 0).
Subdividing this 12-gon into 12 isosceles triangles, we have a polygon with area
$12\left(\dfrac{1}{2}\cdot 5\cdot 5\cdot \sin 30°\right)=12\cdot\dfrac{25}{4}=\underline{\mathbf{75}}$.
Here is a to-scale diagram of the 12-gon and the multiplier R on the unit circle drawn in the complex plane.





B) The algebra is much easier, if we designate the sides as $x-2,\ x,\ x+2$, *but remember the answer will then be* $\boxed{x-2}$. The area of the shaded region is $(x-2)^2+x^2+(x+2)^2=3x^2+8$.
The area of the unshaded region is $(3x)^2-\left(3x^2+8\right)=6x^2-8$.

Thus, the required condition is $\dfrac{6x^2-8}{3x^2+8}>1.9=\dfrac{19}{10}\Leftrightarrow 60x^2-80>57x^2+152$

$3x^2>232\Rightarrow x^2>77^+\Rightarrow x\ge 9$. Thus, the shortest possible side-length for the smallest square is $\underline{\mathbf{7}}$.

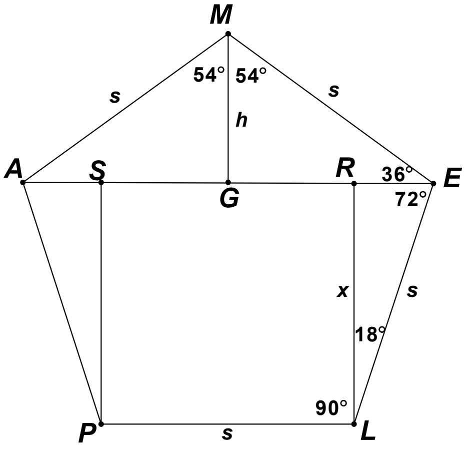
FYI:

Check: For 6, $\dfrac{(6+8+10)^2-(36+64+100)}{(36+64+100)}=\dfrac{576-200}{200}=\dfrac{376}{200}=\dfrac{188}{100}=1.88$ which fails.

Confirmation: For 7, $\dfrac{(7+9+11)^2-(49+81+121)}{(49+81+121)}=\dfrac{729-251}{251}=\dfrac{478}{251}\overset{?}{>}\dfrac{19}{10}=\dfrac{20-1}{10}$

Cross-multiplying, since $4780>5020-251=4769$, the original inequality was true!

Team Round - continued



C) The angles in a regular pentagon measure 108°.
Pentagon *MAPLE*, diagonal $\overline{AE}$, the given altitudes, and the measures of a few selected angles are given in the diagram at the right, as well as variables to represent the lengths of some of the segments.
The area of ΔMAG is *half* the area of ΔMAE, which can be represented as $\frac{1}{2}s^2\sin 108°$. The area of *PLRS* is *sx*.

In ΔLRE, $\sin 72° = \frac{x}{s}$. Thus, $sx = s^2 \sin 72°$

The required ratio is $\dfrac{\frac{1}{4}s^2\sin 108°}{s^2\sin 72°} = \dfrac{\sin 108°}{4\sin 72°}$.
But since angles with measures of 108° and 72° are supplementary, $\sin 108° = \sin 72°$, and the required ratio is **<u>1 : 4</u>**.

D) Multiplying by 10^N moves the decimal point *N* places to the right
To evaluate $\left(0.\overline{2013} + 0.\overline{11011}\right)$ we note that the sum will repeat after 20 places, since the repeating cycles are 4 and 5, and the LCM(4, 5) = 20.

$$\begin{array}{r} .20132\,01320\;13201\;32013 \\ +\;.11011\;11011\;11011\;11011 \\ \hline .\underline{\underline{31143}}\;\;\underline{12331}\;\;\underline{\underline{24212}}\;\underline{43024} \end{array}$$

Since the only zero is in the 18th decimal place, if 10 is raised to the 17th power, the digit in the tenth's place will be a zero. In fact, any exponent of the form 17 + 20k will insure a zero digit in the tenth's place. $2n^2 + n - 28 = 17 + 20k \Leftrightarrow 2n^2 + n - (45 + 20k) = 0$.
Applying the Q.F., we require the discriminant be a perfect square. $1 + 8(45 + 20k) = 361 + 160k$
$k < 0$ fails to produce a perfect square, and eventually produces a negative discriminant.

$$k = 0 \Rightarrow \frac{-1 \pm \sqrt{19^2}}{4} = \frac{-1 \pm 19}{4} = \xcancel{\frac{9}{2}}, \underline{\mathbf{-5}} \qquad \begin{array}{l} k = 1 \Rightarrow 521 \\ k = 2 \Rightarrow 681 \end{array} \qquad k = 3 \Rightarrow 841 = 29^2 \Rightarrow \frac{-1 \pm 29}{4} = \underline{\mathbf{7}}, -\xcancel{\frac{15}{2}}$$

<u>FYI</u>: $k = 14 \Rightarrow \dfrac{-1 \pm 51}{4} = \xcancel{\dfrac{25}{2}}, -13$ $\qquad k = 21 \Rightarrow \dfrac{-1 \pm 61}{4} = -\xcancel{\dfrac{31}{2}}, 15$

(a smaller negative value) (a larger positive value)

Subsequent values of *k* will alternately produce smaller negatives and larger positives.
Thus, $(L, S) = \underline{(\mathbf{-5, 7})}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019 SOLUTION KEY

Team Round - continued

E) $\sin^2(165° - x) + \cos^2(2x - 195°) = 1$

$\Leftrightarrow \sin^2(165° - x) = 1 - \cos^2(2x - 195°) = \sin^2(2x - 195°)$

Therefore,

- the arguments are equal or differ by a multiple of 180°
  same or opposite quadrants (1st and 3rd or 2nd and 4th).
  In the latter case, the sine functions will have opposite signs, but both sides of the equation are being squared.
- the arguments sum to 0° (plus a multiple of 180°) - adjacent quadrants
  (1st and 4th , 2nd and 3rd , 1st and 2nd, or 3rdand 4th).
  The sine functions will have the opposite signs in the first two cases and the same sign in the last two cases, but both sides of the equation are being squared.

Symbolically, $\sin^2 A = \sin^2 B \Leftrightarrow \begin{cases} A = B + n(180°) \\ A + B = 0° + n(180°) \end{cases}$.

Case 1: same or opposite quadrants

$(165° - x) = 2x - 195° + n(180°)$

$\Rightarrow 3x = 360° - n(180°)$

$\Rightarrow x = 120° - n(60°)$.

$n = 1, \ldots, -3 \Rightarrow x = \underline{\mathbf{60°, 120°, 180°, 240°, 300°}}$.

Case 2: adjacent quadrants

$(165° - x) + (2x - 195°) = 0° + n(180°)$

$\Rightarrow x = 30° + n(180°)$

$n = 0, 1 \Rightarrow x = \underline{\mathbf{30°, 210°}}$.

Checks:

$30 \Rightarrow 135$ vs. -135 (45° family in quadrants 2, 3).

$60 \Rightarrow 105$ vs. -75 (75° family in quadrants 2, 4).

$120 \Rightarrow 45$ vs. 45 .

$180 \Rightarrow -15$ vs. 165 (15° family in quadrants 2, 4).

$210 \Rightarrow -45$ vs. 225 (45° family in quadrants 3, 4).

$240 \Rightarrow -75$ vs. 285 (75° family - coterminal in quadrant 4).

$300 \Rightarrow -135$ vs. 405 (45°) (45° family in quadrants 1, 3).

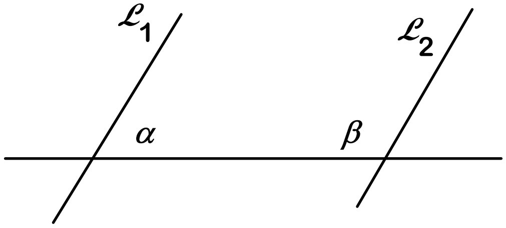
Team Round - continued

F) Interior angles on the same side of a transversal, must be supplementary to insure that the lines crossing the transversal are parallel.

$k(10y+x)+10x+y=180 \Leftrightarrow (10+k)x+(10k+1)y=180$.

Clearly, k = 1 fails.

For k = 2, we would require



$$12x+21y=180 \Leftrightarrow 4x+7y=60 \Leftrightarrow x=\frac{60-7y}{4}=15-y-\frac{3y}{4} \Rightarrow y=\cancel{0},4,8 \Rightarrow x=8,1.$$

Thus, $(k,x,y)=\underline{\mathbf{(2,1,8)}},\underline{\mathbf{(2,8,4)}}$ (so far).

Check: $18+2(81)=180,\ 84+2(48)=180$.

$k=3 \Rightarrow 13x+31y=180$ has solutions, e.g., $(x,y)=(-10,10)$,
but <u>none</u> for which $1 \le x \le 9$ and $1 \le y \le 9$.

$k=4 \Rightarrow 14x+41y=180 \Rightarrow (x,y)=(7,2) \Rightarrow \underline{\mathbf{(4,7,2)}}$. Check: $72+4(27)=180$.

$$k=5,6,7 \Rightarrow \begin{cases} 15x+51y=180 \\ 16x+61y=180 \\ 17x+71y=180 \end{cases}$$, all of which fail.

$k=8 \Rightarrow 18x+81y=180 \Rightarrow 2x+9y=20 \Rightarrow (x,y)=(1,2) \Rightarrow \underline{\mathbf{(8,1,2)}}$.

Check: $12+8(21)=180$.

$k=9 \Rightarrow 19x+91y=180$ which fails.

Thus, our search turns up exactly 4 triples $\underline{\mathbf{(2,1,8),\ (2,8,4),\ (4,7,2),\ (8,1,2)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2019 ANSWERS

Round 1 Algebra 2: Complex Numbers

A) 4

B) $-\frac{3}{8}$ (or -0.375)

C) $-\frac{81}{256}$ (or -0.31640625)

Round 2 Algebra 1: Anything

A) (7, 323)

B) 10

C) 36

Round 3 Geometry: Area

A) 1 : 2

B) (12, 40)

C) $2+\sqrt{2}$

Round 4 Algebra 1: Factoring

A) 301

B) 96

C) 9

Round 5 Trig: Special Angles

A) 30

B) $-\frac{2}{3}$

C) 3

Round 6 Geometry: Angles, Triangles, and Parallels

A) 138, 174
(order irrelevant)

B) 70, 77, 82, 131
(in this order)
Answers must *not*
include the degree symbol.

C) $(19,8)$, $(103,26)$

Team Round

A) 75

B) 7

C) 1 : 4

D) $(-5, 7)$

E) 30°, 60°, 120°, 180°, 210°, 240°, 300°
(degree symbols are not necessary)

F) $(2,1,8)$, $(2,8,4)$, $(4,7,2)$, $(8,1,2)$