MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019
ROUND 1 ALGEBRA 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) (________ , ________ , ________)

A) Distinct values are assigned without repetition to *a*, *b*, *c*, and *d* from the solutions of the equation $(x+1)^2(x+2)^4(4-x)^6(1+2x)^8=0$.

Compute the *maximum* value of the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

B) The determinant $D=\begin{vmatrix} x & 4-3x \\ 5+x & 2x-1 \end{vmatrix}$ has a *minimum* value of *k* for $x=h$.

Compute the ordered pair (h,k).

C) For some integer value of *k*, the system $\begin{cases} (k-1)x+ky=9 \\ \dfrac{x}{2(6-k)}+\dfrac{y}{6-5k}=4 \end{cases}$ represents a pair of perpendicular lines that intersect at the point $P(m,n)$. Compute the ordered triple (k,m,n).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019 SOLUTION KEY

Round 1

A) Values for a, b, c, and d are selected from $\left\{-1,-2,4,-\frac{1}{2}\right\}$.

Since, by definition, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$, we want the value of the product bc to be the negative product with the largest absolute value. There are 24 possible arrangements which produce six different values, specifically 3 pairs of opposites. $\begin{vmatrix} -1 & -2 \\ 4 & -\frac{1}{2} \end{vmatrix} = \underline{\mathbf{8.5}}$ or $\underline{\mathbf{\frac{17}{2}}}$ is largest possible value.

$\begin{vmatrix} -1 & -2 \\ 4 & -.5 \end{vmatrix} = 8.5$, $\begin{vmatrix} -2 & 4 \\ -1 & -.5 \end{vmatrix} = 5$, $\begin{vmatrix} -1 & 4 \\ -.5 & -2 \end{vmatrix} = 4$, $\begin{vmatrix} -.5 & -2 \\ -1 & 4 \end{vmatrix} = -4$. As shown by the last pair of determinants, interchanging the two rows produces an opposite value. 24 possible arrangements producing 6 different values implies that each value can be produced in 4 different ways.

The other determinants that evaluate to 4 are $\begin{vmatrix} -1 & -.5 \\ 4 & -2 \end{vmatrix}$, $\begin{vmatrix} -2 & 4 \\ -.5 & -1 \end{vmatrix}$, and $\begin{vmatrix} -2 & -.5 \\ 4 & -1 \end{vmatrix}$.

B) $D = \begin{vmatrix} x & 4-3x \\ 5+x & 2x-1 \end{vmatrix} = x(2x-1)-(5+x)(4-3x)$

$= (2x^2-x)-(20-11x-3x^2) = 5x^2+10x-20 = 5(x^2+2x-4) = 5\left((x+1)^2-5\right)$

The minimum value of x occurs for $x=-1$. $\therefore (h,k) = \underline{(\mathbf{-1,-25})}$.

C) Given: $\begin{cases} (k-1)x+ky=9 \\ \dfrac{x}{2(6-k)}+\dfrac{y}{6-5k}=4 \end{cases} \Rightarrow \begin{cases} m_1 = \dfrac{1-k}{k} \\ m_2 = \dfrac{6-5k}{2(k-6)} \end{cases}$

For non-vertical perpendicular lines, the product of the slopes must be **-**1.

$$\frac{1-k}{k}\cdot\frac{6-5k}{2(k-6)} = -1 \Leftrightarrow 6-5k-6k+5k^2 = -2k^2+12k \Leftrightarrow 7k^2-23k+6 = (k-3)(7k-2) = 0.$$

$$k=3 \Rightarrow \begin{cases} 2x+3y=9 \\ \dfrac{x}{6}-\dfrac{y}{9}=4 \end{cases} \Leftrightarrow \begin{cases} 6x+9y=27 \\ 6x-4y=144 \end{cases} \Rightarrow 13y=-117 \Rightarrow (x,y)=(18,-9)$$

Therefore, $(k,m,n) = \underline{(\mathbf{3,18,-9})}$.

<u>FYI</u>: Except for the integer restriction on k, $k=\frac{2}{7}$ produces a valid pair of perpendicular lines, namely, $\begin{cases} 5x-2y=-63 \\ 14x+35y=40 \end{cases}$, which intersect at $(m,n) = \left(\frac{-2125}{203}, \frac{1082}{203}\right)$.

Using a matrix solution avoids much of the nasty arithmetic. Ask your coach or a teammate!

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019
ROUND 2 ALGEBRA 1: EXPONENTS AND RADICALS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Evaluate: $$\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{2}{3}\right)^2}-\left(\frac{1}{2}+\frac{2}{3}\right)$$

B) Given: $\sqrt{256x^3+13}=11$
For real values of x, compute <u>all</u> possible real values of $\sqrt{80x^2-9}$.

C) Compute <u>all</u> ordered pairs of integers (A,B) for which the irrational real number $\sqrt{17-12\sqrt{2}}$ may be simplified to $A+B\sqrt{2}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019 SOLUTION KEY

Round 2

A) $\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{2}{3}\right)^2}-\left(\frac{1}{2}+\frac{2}{3}\right)=\sqrt{\frac{1}{4}+\frac{4}{9}}-\frac{3+4}{6}=\sqrt{\frac{9+16}{36}}-\frac{7}{6}=\sqrt{\frac{25}{36}}-\frac{7}{6}=-\frac{2}{6}=\underline{\mathbf{-\frac{1}{3}}}$.

B) $\sqrt{256x^3+13}=11 \Rightarrow 256x^3+13=121 \Rightarrow x^3=\frac{108}{256}=\frac{27}{64} \Rightarrow x=\frac{3}{4}$.

$\sqrt{80\left(\frac{3}{4}\right)^2-9}=\sqrt{80\left(\frac{9}{16}\right)-9}=\sqrt{45-9}=\sqrt{36}=\underline{\mathbf{6}}$.

C) Squaring both sides, $\left(\sqrt{17-12\sqrt{2}}\right)^2=\left(A+B\sqrt{2}\right)^2 \Rightarrow 17-12\sqrt{2}=\left(A^2+2B^2\right)+2AB\sqrt{2}$

Equating the rational and irrational components, we have $\begin{cases} A^2+2B^2=17 \\ AB=-6 \end{cases}$

Both $(3,-2)$ and $(-3,2)$ satisfy the system of equations, but since $\sqrt{17-12\sqrt{2}}$ must be nonnegative, we must determine which ordered pair produces a nonnegative difference.

Is $3>2\sqrt{2}$ or is $3<2\sqrt{2}$? Since the values on both sides of the inequality are positive, squaring both sides is allowed. Testing the first inequality, we have $3^2>\left(2\sqrt{2}\right)^2 \Leftrightarrow 9>8$.

Since the latter is true, $3>2\sqrt{2} \Rightarrow 3-2\sqrt{2}>0$, and the only ordered pair satisfying the required conditions is $\underline{\mathbf{(3,-2)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019
ROUND 3 TRIGONOMETRY: ANYTHING

ANSWERS

A) ________________________________

B) (_____________ , ______________)

C) ________________________________

A) Point P lies on the horizontal line $y = \operatorname{Sin}^{-1}(a) + \operatorname{Cos}^{-1}(b)$, where a and b are constants in the respective domains of these inverse *functions*.
Compute the maximum distance from P to the x-axis.

B) Given: $y = -2\sin\left(2x + \frac{\pi}{2}\right) + 2$
For a minimum positive value $x = h$, this function has a maximum value of k.
Compute the ordered pair (h, k).

C) Solve over $0 \le \theta < 2\pi$: $\qquad \dfrac{\sqrt{1-\sin\theta}}{\sqrt{2}\cos\theta} = 1$

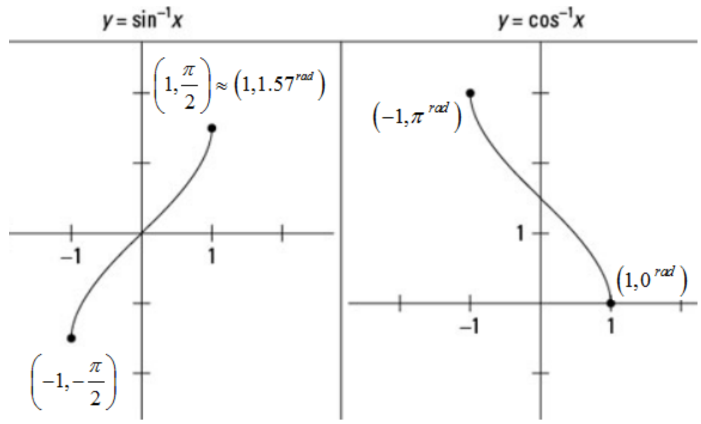
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019 SOLUTION KEY

Round 3

A) The range of the principle inverse sine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\text{Sin}^{-1}(1) = \frac{\pi}{2}$.

The range of the principle inverse cosine function is $[0, \pi]$ and $\text{Cos}^{-1}(-1) = \pi$.

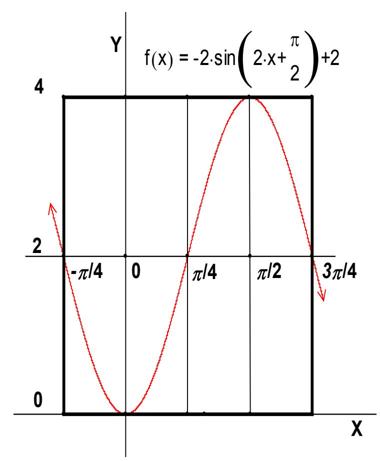
Thus, the maximum distance occurs for $a = 1$ and $b = -1$ namely, $\underline{\boldsymbol{\frac{3\pi}{2}}}$.

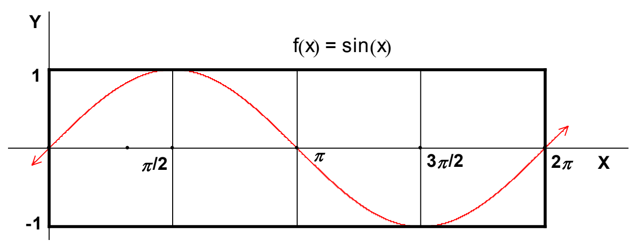


B) Since $y = \sin x$ has a period of 2π, we have $0 \le 2x + \frac{\pi}{2} < 2\pi \Rightarrow -\frac{\pi}{4} \le x < \frac{3\pi}{4}$.

Thus, the "standard" view window of one period of this cyclical function has shifted to the left $\frac{\pi}{4}$ units (phase shift), shifted up 2 units (vertical displacement), compressed by a factor of 2 in the x-direction (period changed to π units , from $-\frac{\pi}{4}$ to $\frac{3\pi}{4}$), and expanded by a factor of 2 in the y-direction (change in amplitude 2 above and 2 below a center line of $y = 2$). Also, the factor of -2 has inverted the graph; hills have become valleys, and vice versa.

$(h, k) = \left(\frac{\pi}{2}, 4\right)$.





C) Given: $\dfrac{\sqrt{1-\sin\theta}}{\sqrt{2}\cos\theta} = 1$

Note: θ is restricted to quadrant 1 or 4 to insure the quotient on the left side of the equation is positive. Squaring both sides, $1 - \sin\theta = 2\cos^2\theta = 2\left(1 - \sin^2\theta\right)$

$\Leftrightarrow 2\sin^2\theta - \sin\theta - 1 = 0 \Leftrightarrow (2\sin\theta + 1)(\sin\theta - 1) = 0$

$\Rightarrow \sin\theta = -\frac{1}{2}, 1 \Rightarrow \theta = \cancel{\frac{7\pi}{6}}$ (Q3), $\frac{11\pi}{6}, \frac{\pi}{2}$. Checking the quadrantal value,

$\cos\frac{\pi}{2} = 0 \Rightarrow$ division by zero $\Rightarrow \frac{\pi}{2}$ is also extraneous. Thus, the solution $\underline{\boldsymbol{\frac{11\pi}{6}}}$ is unique.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019
ROUND 4 ALGEBRA 1: ANYTHING

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Given: a ✎ $b=(a+b)+ab$, a ✒ $b=(a-b)+\dfrac{a}{b}$

Compute <u>all</u> values of x for which x ✒ $4 = 2$ ✎ 3.

B) In the puzzle below, each card shows a digit or hides a digit. *Hidden cards* $\left(\boxed{\otimes}\right)$ *are not necessarily the same*. The three-digit number on the left side of the equation is a multiple of 3. The three-digit number on the right side is a multiple of 11. Determine <u>all</u> possible digits which could be on the hidden single digit card on the left side of the equation.

$$\boxed{6}\boxed{\otimes}\boxed{4} + \boxed{\otimes} = \boxed{\otimes}\boxed{\otimes}\boxed{\otimes}$$

C) On a true/false test, Claude, who is always totally unprepared, guessed at the answers and managed to get 22 out of the first 85 questions correct. Thereafter, his luck improved, and he managed to answer 2 out of every 5 questions correctly. Overall, he answered 30% of the questions correctly. If he had studied, he could have answered all the remaining questions correctly, improving his percentage of correctly answered questions by k percentage points. Compute k.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019 SOLUTION KEY

Round 4

A) Given: a ✐ $b = (a+b) + ab$, a ✒ $b = (a-b) + \frac{a}{b}$

x ✒ $4 = 2$ ✐ $3 \Leftrightarrow (x-4) + \frac{x}{4} = (2+3) + 2 \cdot 3 = 11$

$\Rightarrow 4x - 16 + x = 44 \Rightarrow 5x = 60 \Rightarrow x = \underline{\mathbf{12}}$.

B) $\boxed{❻}\boxed{⊗}\boxed{❹}$ + $\boxed{⊗}$ = $\boxed{⊗}\boxed{⊗}\boxed{⊗}$

The sum of the digits of a multiple of 3 must also be a multiple of 3.

On the left the sum of the digits in $\boxed{❻}\boxed{⊗}\boxed{❹}$ is $10 + n \Rightarrow n = 2, 5$, or 8.

Let ☺ denote the single digit in the sum of the left side.

Since $11\overline{)624}^{\,56}$ with a remainder of 8, ☺ = $\underline{\mathbf{3}}$ produces a sum (627) which is a multiple of 11.

Since $11\overline{)654}^{\,59}$ with a remainder of 5, ☺ = $\underline{\mathbf{6}}$ produces a sum (660) which is a multiple of 11.

Since $11\overline{)684}^{\,62}$ with a remainder of 2, ☺ = $\underline{\mathbf{9}}$ produces a sum (693) which is a multiple of 11.

<u>FYI</u>: Divisibility Test for 11.
Let N be an integer with n digits, where n is any natural number, i.e., $N = d_1 d_2 d_3 d_4 .... d_n$.
An integer N is divisible by 11 if and only if the difference between the sum of the digits in even positions $d_2 + d_4 + d_6 + ...$ and the sum of the digits in odd positions $d_1 + d_3 + d_5 + ...$ is a multiple of 11. Remember: 0 is a multiple of 11.

C) Assume there are $5x$ additional questions.

$\frac{22+2x}{85+5x} = \frac{3}{10} \Rightarrow 220 + 20x = 255 + 15x \Rightarrow 5x = 35 \Rightarrow x = 7$

Therefore, there were 120 questions on the test.
If he answered all the remaining 35 questions correctly, he would have answered 57 out of 120 questions correctly $\Rightarrow \frac{57}{120} = \frac{19}{40} = 47.5\%$ correct, an improvement of 17.5 percentage points. $\therefore k = \underline{\mathbf{17.5}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019
ROUND 5 PLANE GEOMETRY: ANYTHING

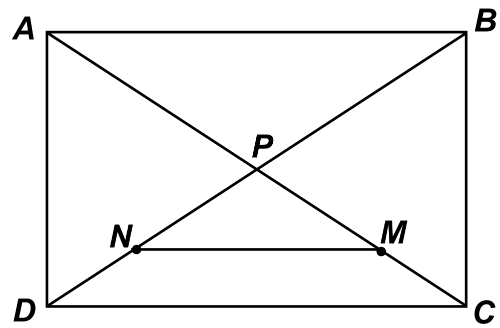
ANSWERS

A) ________________________________

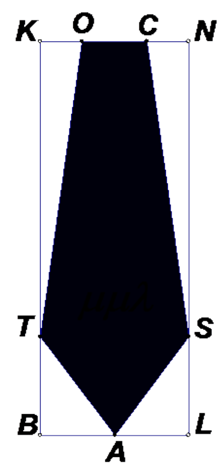
B) ________________________________

C) ________________________________

A) Rectangle $ABCD$ is 6 units by 8 units.
The diagonals intersect at point P.
Point M lies on $\overline{PC}$ and $PM : MC = 2:1$
N lies on $\overline{PD}$ such that $\overline{MN} \parallel \overline{CD}$.
Compute the area of ΔPMN.



B) Given: $BL = 6$, $BK = 16$, $AB = AL$, $KO = NC$, $KT = NS$, and
$KT : TB = 3:1$
In the diagram at the right, the area of pentagon $ASCOT$ is at least 60% of the area of rectangle $BLNK$. Compute the <u>maximum</u> value of NC.



C) If the number of sides in polygon P is increased by 50%, the number of diagonals increases by 140%. Compute the number of sides in polygon P.

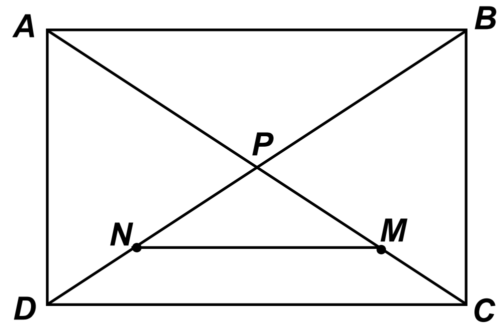
Note:

A 100% increase is equivalent to doubling the original quantity.

A 140% increase is equivalent to adding 1.4 times the original quantity to the original quantity, or multiplying the original quantity by 2.4 .

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019 SOLUTION KEY

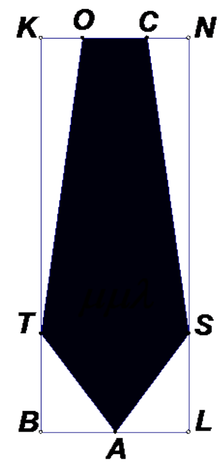
Round 5



A) The area of ΔPCD is 12 units^2. $\dfrac{PN}{PD}=\dfrac{2}{3}$.

$\Delta PNM \sim \Delta PDC \Rightarrow$ ratio of areas is 4 : 9.

Thus, the area of ΔPMN is $\dfrac{4}{9}(12)=\underline{\mathbf{\dfrac{16}{3}}}$.



B) $BK=16$, $KT:TB=3:1\Rightarrow(KT,TB)=(12,4)$

If $OC=2x$, then $NC=KO=3-x$ and the area of $ASCOT$ is $96-\left(12(3-x)+12\right)=48+12x$.

Then:

$48+12x\geq 0.6(96)=57.6\Rightarrow x\geq\dfrac{9.6}{12}=0.8\Rightarrow NC_{\max}=3-0.8=\underline{\mathbf{2.2}}$.

C) If we let $2k$ denote the number of sides in the original polygon, then $3k$ would represent the number of sides in the new polygon.
Thus,

$$2.4\left(\frac{2\cancel{k}(2k-3)}{\cancel{2}}\right)=\frac{3\cancel{k}(3k-3)}{\cancel{2}}\Leftrightarrow\frac{12}{5}(4k-6)=9k-9\Leftrightarrow 48k-72=45k-45\Rightarrow 3k=27\Rightarrow k=9$$

Original polygon: **<u>18</u>** sides (135 diagonals)

[Checking - New polygon: 27 sides (324 diagonals) and 2.4(135) = 324]

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019
ROUND 6 ALGEBRA 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) (_____________ , _____________)

B) ______________________________

C) ______________________________

A) A box contains 5 red marbles, 6 white marbles and 9 blue marbles. Two marbles are taken at random from this box without replacement. The probability that at least one of the selected marbles is red is $\frac{A}{B}$, a reduced fraction. Determine the ordered pair (A, B).

B) Compute the sum of the coefficients of the x^4y^3-term and the x^3y^4-term in the expansion of $\left(4x+\frac{y}{2}\right)^7$.

C) Given points $A(0, 8)$, $B(-6, 0)$ and $C(6, 0)$. Compute the probability that an arbitrary point in the interior of ΔABC will be closer to the vertex of the vertex angle of this isosceles triangle than the vertex of either base angle.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019 SOLUTION KEY

Round 6

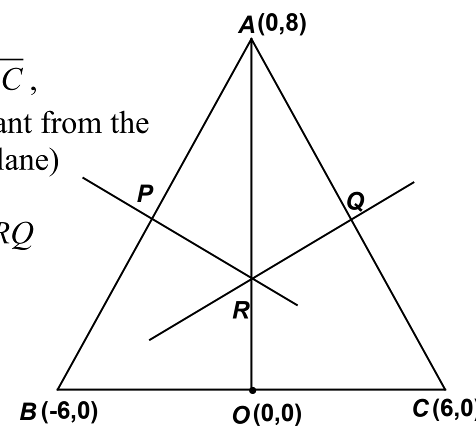
A) At least one red would be equivalent to 1, 2, 3, 4, or 5 red. To avoid 5 separate cases, look at the problem indirectly. Consider two mutually exclusive events which include all possible outcomes. The probability that either one or the other occurs is 100%. Therefore, the probability that at least one marble is red is equivalent to 1 minus the probability that none of the marbles are red.

Indirect Rule: P(success) = 1 – P(failure)

P(at least 1 red) = $1-\frac{15}{20}\cdot\frac{14}{19}\cdot=1-\frac{3}{\cancel{4}_{2}}\cdot\frac{7}{19}=1-\frac{21}{38}=\frac{17}{38}\Rightarrow(A,B)=\underline{(\mathbf{17,38})}$.

B) The required sum is

$$\binom{7}{3}4^4\left(\frac{1}{2}\right)^3+\binom{7}{4}4^3\left(\frac{1}{2}\right)^4=\frac{35\cdot 256}{8}+\frac{35\cdot 64}{16}=35\cdot 32+35\cdot 4=35\cdot 36=\underline{\mathbf{1260}}.$$



C) Let $\overleftrightarrow{PR}$ and $\overleftrightarrow{QR}$ be the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$, respectively. Points on a perpendicular bisector are equidistant from the endpoints of the bisected segment. All other points (in the plane) are closer to one endpoint than the other.
The "desirable" points are in the interior of quadrilateral *APRQ* which must be a kite.
Since the diagonals of a kite are perpendicular, the required probability is

$$\frac{\text{area}(APRQ)}{\text{area}(ABC)}=\frac{\frac{1}{2}AR\cdot PQ}{48}=\frac{AR\cdot PQ}{96}.$$

$P(-3,4)$ and $Q(3,4)\Rightarrow PQ=6$.

$$m_{AB}=\frac{4}{3}\Rightarrow m_{\perp}=-\frac{3}{4}\Rightarrow PR:3x+4y=7\Rightarrow R\left(0,\frac{7}{4}\right)\Rightarrow AR=8-\frac{7}{4}=\frac{25}{4}$$

$$\Rightarrow\frac{6\cdot\frac{25}{4}}{96}=\underline{\frac{\mathbf{25}}{\mathbf{64}}}$$ (or **0.390625** or **39.0625%** **or** $\underline{\mathbf{39\frac{1}{16}\%}}$).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019
ROUND 7 TEAM QUESTIONS

ANSWERS

A) (____ , ____ , ____ , ____) D) (____ , ____ , ____ , ____ , ____ , ____)

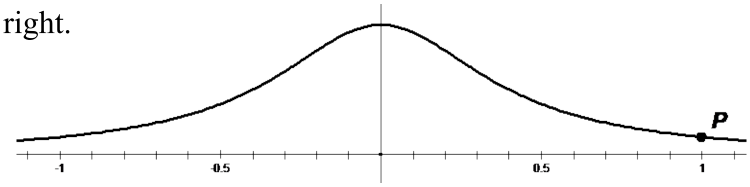
B) ______________________________ E) ______________________________

C) ______________________________ F) ______________________________

A) Two points determine a line and 3 *non-collinear* points determine a plane.
The three points (2, –1, 0), (3, 0, 2) and (5, 1, -2) determine a plane whose equation is $Ax + By + Cz + D = 0$. If A, B, C, and D are integers with no common factor (other than 1), and $A > 0$, determine the ordered 4-tuple (A, B, C, D).

B) For integer constants m, the system $\begin{cases} x + my = 2019 \\ mx + y = 2019m + 8 \end{cases}$ has exactly three solutions over the integers; specifically, $(x_1,\ y_1), (x_2,\ y_2), (x_3,\ y_3)$. Compute the quotient $\dfrac{x_1y_1 + x_2y_2 + x_3y_3}{6}$.

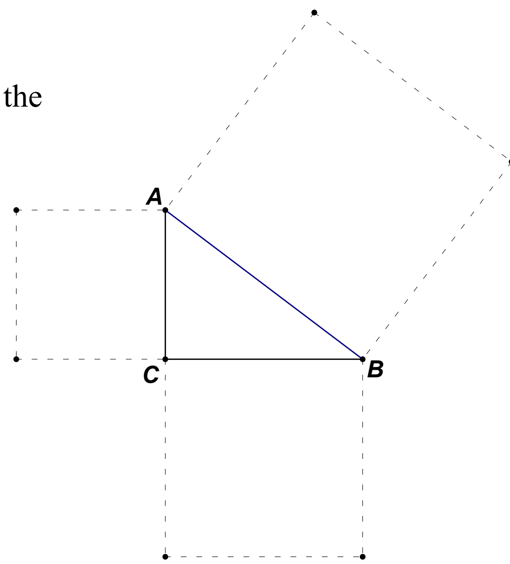
C) The graph of $\begin{cases} x = a\cot\phi \\ y = a\sin^2\phi \end{cases}$ is shown at the right.
It is called the "Witch of Agnesi".
If $(x, y) = \left(1, \dfrac{8}{145}\right)$ is a point on the graph, compute a, given that $a = \dfrac{k}{10}$, for some integer k.
Note: *The scales on the x-axis and y-axis are* _not_ *the same.*



D) The hardwood tongue and groove planking I want for the floor in my study comes in 12-foot lengths. The exposed width of each plank is 6 inches. My study is 8 feet 6 inches by 14 feet. Alternating rows will use either one 8-foot plank and one 6-foot plank or two 4-foot planks and one 6-foot plank. Suppose I buy k 12-foot planks and cut them into lengths I can use. For n, a *minimum* number of cuts, I can cover my floor with a 4-foot lengths, b 6-foot lengths, and c 8-foot lengths. For the ordered triple (a,b,c), I wasted w feet of planking. Compute the ordered 6-tuple (k,w,n,a,b,c).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019
ROUND 7 TEAM QUESTIONS

E) In right ΔABC, $AC = 6$, $BC = 8$ and $\overline{AC} \perp \overline{BC}$.
Let P, Q and R be the centers of the squares drawn on the sides of ΔABC. Compute the area of ΔPQR.



F) My wife and I and our three kids were visiting family in rural upstate New York on a very hot summer day. We were all craving some ice cream, so we stopped at a convenience store. The only flavors available were strawberry, chocolate, and vanilla. Compute the number of different *unordered* selections of flavors.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019 SOLUTION KEY

Team Round

A) The plan is to find expressions for A, B and C in terms of D and then pick D, so that the integer and greatest common factor requirements are satisfied.

If $\begin{cases}(2,-1,0)\\(3,0,2)\\(5,1,-2)\end{cases}$ satisfy $Ax+By+Cz+D=0$, then $\begin{cases}(1)\ 2A-B+D=0\\(2)\ 3A+2C+D=0\\(3)\ 5A+B-2C+D=0\end{cases}$.

(1) $\Rightarrow A=\dfrac{B-D}{2}$

(2) $\Rightarrow C=\dfrac{-3A-D}{2}=\dfrac{-3\left(\dfrac{B-D}{2}\right)-D}{2}=\dfrac{-3B+D}{4}$

(3) $\Rightarrow B=-5A+2C-D=-5\left(\dfrac{B-D}{2}\right)+2\left(\dfrac{-3B+D}{4}\right)-D \Leftrightarrow$

$\Leftrightarrow 2B=-5B+5D-3B+D-2D \Leftrightarrow 10B=4D \Leftrightarrow \boxed{B=\frac{2}{5}D}$

Substituting in (1), $A=\dfrac{\frac{2}{5}D-D}{2} \Rightarrow \boxed{A=-\frac{3}{10}D}$

Substituting in (2), $C=\dfrac{-\frac{6}{5}D+D}{4} \Rightarrow \boxed{C=-\frac{D}{20}}$

To ensure that A, B, C, and D are integers, D must be at least 20.
To ensure that A, B, C, and D have no common factor, D must be at most 20. Thus, $D = 20$.
Since $D=20$ produces a negative A-value, we use $D=-20$ and $(A,B,C,D)=\underline{\mathbf{(6,-8,1,-20)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019 SOLUTION KEY

Team Round - continued

B) $\begin{cases} x+my=2019 \\ mx+y=2019m+8 \end{cases} \Leftrightarrow \begin{cases} mx+m^2y=2019m \\ mx+y=2019m+8 \end{cases} \Rightarrow y\left(1-m^2\right)=8 \Rightarrow y=\dfrac{8}{1-m^2}$.

Thus, we have integer values for y when $m=0,\ 3,\ -3$, namely, $y=8,\ -1,\ -1$.

Substituting for y in the first equation, $x=2019-\dfrac{8m}{1-m^2}$

$\Rightarrow (x,y)=(2019,8),\ (2022,-1),\ (2016,-1) \Rightarrow$

$\dfrac{x_1y_1+x_2y_2+x_3y_3}{6}=\dfrac{2019\cdot 8-2022-2016}{6}=\dfrac{2019(8-2)}{6}=\underline{\mathbf{2019}}$.

C) The plan is to eliminate the parameter ϕ and express y directly in terms of x.

$x=a\cot\phi \Rightarrow x^2+a^2=a^2\cot^2\phi+a^2=a^2\left(\cot^2\phi+1\right)=a^2\csc^2\phi$.

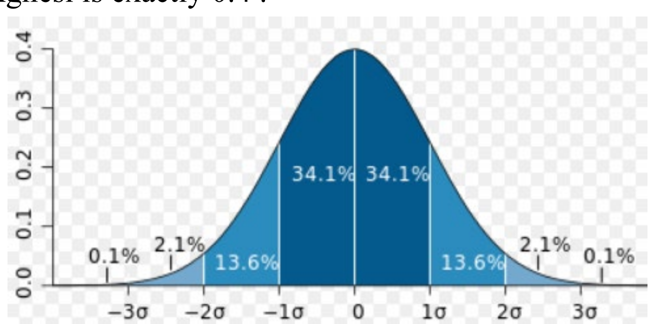
$y=a\sin^2\phi \Rightarrow y=\dfrac{a}{\csc^2\phi}=\dfrac{a^3}{a^2\csc^2\phi}=\dfrac{a^3}{x^2+a^2}$.

Substituting, $\dfrac{8}{145}=\dfrac{a^3}{1+a^2} \Rightarrow 145a^3-8a^2-8=0$. By the rational root theorem, k must be a factor of 8. $k=4 \Leftrightarrow a=\underline{\mathbf{0.4}}$.
Synthetic division confirms 0.4 is a factor.

	145	−8		−8
0.4 |		58	20	8
	145	50	20	0

<u>**FYI**</u>: The graph looks a lot like the graph of the normal probability curve $y=\dfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.

For $x=0$, the maximum height of the Witch of Agnesi is exactly 0.4 .
With a calculator, we can confirm the maximum height of the probability curve is approximately 0.3989 . This height assures that the area "under" the curve is 1, i.e. 100%.
At $x=1$, the probability curve returns a value of 0.2420 (as compared to $\dfrac{8}{145}\approx 0.055$).
Along the x-axis, the symbol σ denotes a standard deviation. The normal probability curve predicts that 68.2% of results will fall within 1 standard deviation of the median value (above or below). (95.4% within 2 standard deviations / 99.6% within 3 standard deviations). Results further from the median are frequently referred to as "outliers".



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019 SOLUTION KEY

Team Round - continued

D) Covering the 8.5-foot by 14-foot space will require either seventeen 14-foot rows (each 6 inches wide). At a minimum, I would need $8.5 \cdot 2 \cdot 14 = 238$ linear feet of planking.
$12k \geq 2 \cdot 14 \cdot 8.5 = 238 \Rightarrow k_{\min} = 20,\ w_{\min} = 2$

Case 1: Start and end with 8-6 rows
We would need 9(8-6 rows) and 8(4-6-4 rows), for a total of 9-8', 17-6', and 16-4'.

	Cut Lengths				Still Needed			Waste
# 12 foot planks	8 foot	6 foot	4 foot	n (# cuts)	8 foot	6 foot	4 foot	0
9	9		9	9	0	17	7	0
2			6	4	0	17	1	0
8		16		8	0	1	1	0
1		1	1	2	0	0	0	2
$k = 20$	$c = 9$	$b = 17$	$a = 16$	$n = 23$				$w = 2$

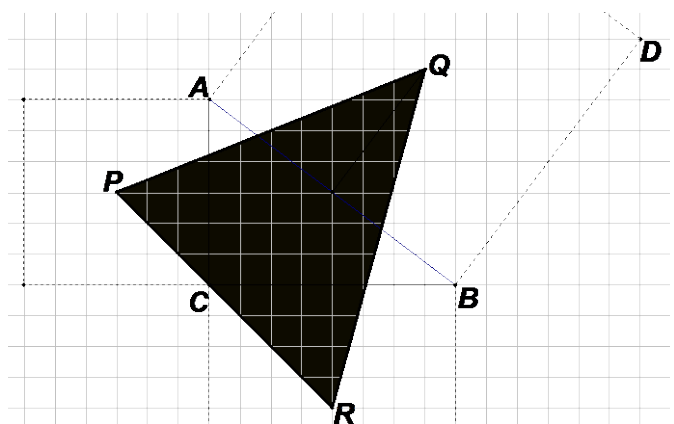
Case 2: Start and end with 4-6-4 rows
We would need 9(4-6-4 rows) and 8(8-6 rows), for a total of 8-8', 17-6', and 18-4'.

	Cut Lengths				Still Needed			Waste
# 12 foot planks	8 foot	6 foot	4 foot	n (# cuts)	8 foot	6 foot	4 foot	0
8	8		8	8	0	17	10	0
8		16		8	0	1	10	0
3			9	6	0	1	1	0
1		1	1	2	0	0	0	2
$k = 20$	$c = 8$	$b = 17$	$a = 18$	$n = 24$				$w = 2$

Thus, case 1 gives the *minimum* number of cuts. $(k, w, n, a, b, c) = \underline{\mathbf{(20, 2, 23, 16, 17, 9)}}$.

Team Round - continued

E) Let ΔABC be placed on the coordinate plane with point C at the origin and point B at (8, 0)
The coordinates of points P and R are $(-3,3)$ and $(4,-4)$. From B to A, $(\Delta x,\ \Delta y) = (-8,6)$.
[Δ (read as delta) is just an abbreviation for "change in the value of"]
Since $\overline{BA} \perp \overline{BD}$, from point B to point D, $(\Delta x,\ \Delta y) = (6,8) \Rightarrow D(14,8)$.
Therefore, Q, the midpoint of $\overline{AD}$ is $(7,7)$.
Using the coordinates of the vertices, the area of ΔPQR is $\frac{1}{2}\left\|\begin{matrix} -3 & 3 & 1 \\ 7 & 7 & 1 \\ 4 & -4 & 1 \end{matrix}\right\|$. Expanding the absolute value of the determinant by column 3, we have $\frac{1}{2}\left\|\begin{matrix} 7 & 7 \\ 4 & -4 \end{matrix}\right| - \left|\begin{matrix} -3 & 3 \\ 4 & -4 \end{matrix}\right| - \left|\begin{matrix} -3 & 3 \\ 7 & 7 \end{matrix}\right\| =$
$\frac{1}{2}\left|-56 - 0 + (-42)\right| = \frac{98}{2} = \underline{\mathbf{49}}$



An alternative: In a vertical column, list the coordinates of the vertices in a CW (or CCW) order, starting at any vertex. Repeat the coordinates of the point with which you started. Subtract the sum of the UPWARD products from the sum of the DOWNWARD products and take half the absolute value of this difference.

$$\frac{1}{2}\left|\begin{matrix} -3 & 3 \\ 7 & 7 \\ 4 & -4 \\ -3 & 3 \end{matrix}\right| = \frac{1}{2}\left|(-3\cdot 7 + 7\cdot -4 + 4\cdot 3) - (-3\cdot -4 + 4\cdot 7 + 7\cdot 3)\right| = \frac{1}{2}\left|-37 - 61\right| = \underline{\mathbf{49}}$$

Alternately, "inscribe" ΔPQR in a rectangle, where R(4, -4) determines the bottom side, P(3, -3) determines the left side, and Q(7, 7) determines the top and right sides. Subtract the area of three right triangles from the area of the rectangle.

$$11\cdot 10 - \left(\frac{1}{2}\cdot 7\cdot 7\right) - \left(\frac{1}{2}\cdot 3\cdot 11\right) - \left(\frac{1}{2}\cdot 4\cdot 10\right) = 110 - \frac{1}{2}(49 + 33 + 40) = 110 - 61 = \underline{\mathbf{49}}.$$

Team Round - continued

F) Solution #1 (Systematic Brute Force)

1 Flavor	2 Flavors		3 Flavors	
	1/4	2/3	1/1/3	1/2/2
CCCCC	C/SSSS	CC/SSS	C/V/SSS	C/VV/SS
VVVVV	S/CCCC	SS/CCC	C/S/VVV	S/CC/VV
SSSSS	V/SSSS	VV/SSS	V/S/CCC	V/SS/CC
	S/VVVV	SS/VVV		
	V/CCCC	VV/CCC		
	C/VVVV	CC/VVV		
3	6	6	3	3
			Total:	<u>**21**</u>

Solution #2: (Formula)

The normal formula for combinations ${}_nC_r = \binom{n}{r} = \dfrac{n!}{r!\cdot(n-r)!}$ is for an *unordered* arrangement of n objects taking r at a time, <u>***without***</u> replacement.
In this case, we must have $n \le r$.

In our situation, r = 5 people are picking 5 flavors from n = 3 available flavors and replacement is a necessity.
The handy formula for combinations <u>*with replacement*</u> is

$$\boxed{\binom{n+r-1}{r}} = \binom{3+5-1}{5} = \binom{7}{5} = \binom{7}{2} = \frac{7\cdot 6}{1\cdot 2} = \underline{\mathbf{21}}.$$

Can you prove this formula is correct?
Please send your argument/explanation/proof to olson.re@gmail.com.
The best explanation(s) received and the name(s) of the submitter(s) will be included in the archival copy of the contest.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2019 ANSWERS

Round 1 Algebra 2: Simultaneous Equations and Determinants

A) 8.5 or $\frac{17}{2}$ B) $(-1,-25)$ C) $(3,18,-9)$

Round 2 Algebra 1: Exponents and Radicals

A) $-\frac{1}{3}$ B) 6 C) $(3,-2)$

Round 3 Trigonometry: Anything

A) $\frac{3\pi}{2}$ B) $\left(\frac{\pi}{2},4\right)$ C) $\frac{11\pi}{6}$ only

Round 4 Algebra 1: Anything

A) 12 B) 3, 6, and 9 C) 17.5

Round 5 Plane Geometry: Anything

A) $\frac{16}{3}$ B) 2.2 C) 18

Round 6 Algebra 2: Probability and the Binomial Theorem

A) $(17,38)$ B) 1260 C) $\frac{25}{64}$

(0.390625 or 39.0625% or $39\frac{1}{16}\%$)

Team Round

A) $(6,-8,1,-20)$ D) $(20, 2, 23, 16, 17, 9)$

B) 2019 E) 49

C) 0.4 F) 21