MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A) _______________________________

B) (________ , ________ , ________)

C) (_____ , _____ , _____ , _____)

A) Given: $f(x)=\begin{cases} 1 & \text{for } x<-\sqrt{5} \\ 0 & \text{for } -\sqrt{5}\le x\le -\sqrt{3} \\ -1 & \text{for } x>-\sqrt{3} \end{cases}$,

$g(x)=f(f(x))$ and $h(x)=f\left(f\left(-\frac{1}{2}\right)\right)$.

Compute $\left[3f(-2)+4g(-3)-2h(-2.5)\right]^3$.

B) If $f(2x)=3x^2-x+2$, then $f(x)=Ax^2+Bx+C$.
Compute the ordered triple (A,B,C).

C) Given: $f(x)=2x^3+x^2-4x-3$
For some <u>integer</u> h, $f(x+h)=Ax^3+Bx^2+C$.
Compute the ordered quadruple (h,A,B,C).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019 SOLUTION KEY

Round 1

A) $f(x)=\begin{cases} 1 & \text{for } x<-\sqrt{5} \\ 0 & \text{for } -\sqrt{5}\le x\le -\sqrt{3} \\ -1 & \text{for } x>-\sqrt{3} \end{cases}$

$\boxed{f(-2)=0}$

For all x, $\boxed{h(x)=f\left(f\left(-\frac{1}{2}\right)\right)=f(-1)=-1}$

$g(x)=f(f(x))\Rightarrow \boxed{g(-3)=f(1)=-1}$,

Substituting, $\left[3f(-2)+4g(-3)-2h(-2.5)\right]^3=\left[3(0)+4(-1)-2(-1)\right]^3=(-2)^3=$ **$\underline{-8}$**.

B) Let $y=2x\Rightarrow x=\frac{y}{2}$. Substituting, $f(y)=3\left(\frac{y}{2}\right)^2-\left(\frac{y}{2}\right)+2=\left(\frac{3}{4}\right)y^2-\left(\frac{1}{2}\right)y+2$

Replacing y with x, $f(x)=\frac{3}{4}x^2-\frac{1}{2}x+2\Rightarrow (A,B,C)=\underline{\left(\mathbf{\frac{3}{4},-\frac{1}{2},2}\right)}$.

C) $f(x+h)=2(x+h)^3+(x+h)^2-4(x+h)-3$
$=2\left(x^3+3x^2h+3xh^2+h^3\right)+\left(x^2+2xh+h^2\right)-4x-4h-3$

Combining like terms,
$2x^3+(6h+1)x^2+\left(6h^2+2h-4\right)x+\left(2h^3+h^2-4h-3\right)$

Thus, $6h^2+2h-4=0\Rightarrow 2(3h-2)(h+1)=0\Rightarrow h=\cancel{\frac{2}{3}},-1$

Substituting, $(h,A,B,C)=\underline{(\mathbf{-1,2,-5,0})}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019
ROUND 2 ARITHMETIC / NUMBER THEORY

ANSWERS

A) ______________________________

B) (____________ , ____________)

C) ______________________________

A) a, b, and c are three consecutive odd positive integers, <u>none</u> of which are prime. Compute the <u>minimum</u> value of $a+b+c$.

B) Compute the ordered pair (a, b), if the eight-digit base ten integer $N = 437213ab$ is divisible by 72.

C) Compute the <u>largest</u> prime factor of $3^{18} - 3^{13} + 3^7 - 1$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019 SOLUTION KEY

Round 2

A) Since all primes are odd (except 2), we are looking for a gap between primes of at least 7.

… **P**, E, **O**, E, **O**, E, **O**, E, **P**, …

Gaps between primes: 3 (7 to 11), 5 (31 to 37), (61 to 67), (73 to 79), (83 to 89), 7 (89 to 97). Testing,

$91 = 7\cdot 13,\ 93 = 3\cdot 31,\ 95 = 5\cdot 19 \Rightarrow a+b+c = 91+93+95 = \underline{\mathbf{279}}$.

B) N must be divisible by 8 and 9.
To guarantee divisibility by 9, the sum of all the digits must be divisible by 9.
To guarantee divisibility by 8, the 3-digit integer $3ab$ must be divisible by 8.
Thus, $20 + a + b$ must equal 27 or 36.
$a + b = 7$ or 16
Since b must be even, we consider $(a, b) = (1, 6), (5, 2), (3, 4)$ and $(8, 8)$.
Only **(5, 2)** satisfies divisibility by 8.

C) Recall: $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Let $(a,b) = (3^6, 1)$.

$3^{18} - 3^{13} + 3^7 - 1 = (3^6)^3 - 3(3^6)^2(1) + 3(3^6)(1)^2 - 1^3 = (3^6-1)^3$

$= (728)^3 = (8\cdot 91)^3 = 2^9 \cdot 7^3 \cdot 13^3$.

Therefore, the largest prime factor is **13**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

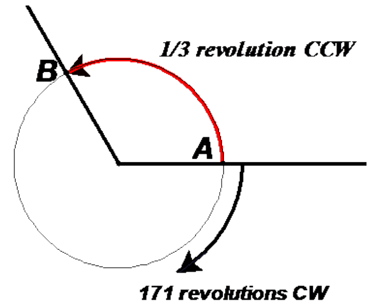
A) Compute the value of N, if N is coterminal with $-\dfrac{1024\pi}{3}$ and $0 \le N < 2\pi$.

B) Simplify completely: $\sin 2x \cdot \sec\left(\dfrac{\pi}{2} + x\right) \cdot \csc\left(\dfrac{\pi}{2} - x\right)$

C) Compute <u>all</u> possible solutions: $Sin^{-1}(4/5) = \pi - 2Tan^{-1}(x)$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019 SOLUTION KEY

Round 3



A) $-\dfrac{1024\pi}{3} = \left(-341\dfrac{1}{3}\right)\pi = \left(-342 + \dfrac{2}{3}\right)\pi$.

-342π is a "dizzy factor", equivalent to 171 clockwise revolutions around the unit circle, returning to the starting point at A. Then: $+\dfrac{2\pi}{3}$ rotates counterclockwise from A to the terminal position B in the second quadrant. Therefore, $N = \underline{\mathbf{\dfrac{2\pi}{3}}}$.

<u>FYI</u>: Turning $+\dfrac{2\pi}{3}$ radians is equivalent to turning counterclockwise through $120° = \dfrac{1}{3}(360°)$.

B) Recall: $\sin 2x = 2\sin x \cos x$

$\dfrac{\pi}{2}+x$: Think *cofunction* / Quadrant 2 (where sec returns a negative value) $\sec\left(\dfrac{\pi}{2}+x\right) = -\csc x$

Alternately, using the expansion for $\cos(A+B)$, $\sec\left(\dfrac{\pi}{2}+x\right) = \dfrac{1}{\cos\left(\dfrac{\pi}{2}+x\right)} = \dfrac{1}{\cos\dfrac{\pi}{2}\cos x - \sin\dfrac{\pi}{2}\sin x} = \dfrac{1}{-\sin x} = -\csc x$

$\dfrac{\pi}{2}-x$: Think *cofunction* / Quadrant 1 (where csc returns a positive value) $\csc\left(\dfrac{\pi}{2}-x\right) = +\sec x$

Alternately, using the expansion for $\sin(A-B)$, $\csc\left(\dfrac{\pi}{2}-x\right) = \dfrac{1}{\sin\left(\dfrac{\pi}{2}-x\right)} = \dfrac{1}{\sin\dfrac{\pi}{2}\cos x - \cos\dfrac{\pi}{2}\sin x} = \dfrac{1}{\cos x} = \sec x$

Therefore, we have $2\sin x\cos x \cdot -\csc x \cdot \sec x = \dfrac{2\sin x \cdot \cos x}{-\sin x \cdot \cos x} = \underline{\mathbf{-2}}$.

Check: Let $x = 30°$ Then:
$\sin 60°/[\cos 120° \cdot \sin 60°] = \sec 120° = -2$.

C) Given: $Sin^{-1}(4/5) = \pi - 2Tan^{-1}(x)$. $A = \text{Tan}^{-1}(x) \Rightarrow \tan(A) = x$ and $\begin{cases} 0 < A < \dfrac{\pi}{2}, \text{ for } x > 0 \\ -\dfrac{\pi}{2} < A < 0, \text{ for } x < 0 \end{cases}$.

Taking the tangent of both sides, $4/3 = \tan(\pi - 2A) = -\tan(2A) \Rightarrow$
$\dfrac{4}{3} = \dfrac{2\tan A}{\tan^2 A - 1} \Rightarrow 4\tan^2 A - 6\tan A - 4 = 0 \Rightarrow 2\tan^2 A - 3\tan A - 2 = (2\tan A + 1)(\tan A - 2) = 0$
$\Rightarrow \tan A = -1/2$ or $+2$. However, the negative value is extraneous, since the right side of the equation will produce a value between $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$, while the left hand will produce a value between 0 and $\dfrac{\pi}{2}$.

Thus, the only solution is <u>**+2**</u>. Alternately, $\text{Tan}^{-1}(x) = \dfrac{\pi}{2} - \dfrac{1}{2}\text{Sin}^{-1}(.8)$. Taking tan of both sides,

$x = \cot\left(\dfrac{1}{2}\text{Sin}^{-1}(.8)\right)$. Since $\cot(x/2) = \dfrac{1+\cos x}{\sin x}$, we have $x = \dfrac{1\ +\ .6}{.8} = \underline{\mathbf{2}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019
ROUND 4 ALG 1: WORD PROBLEMS

ANSWERS

A) ______________________________

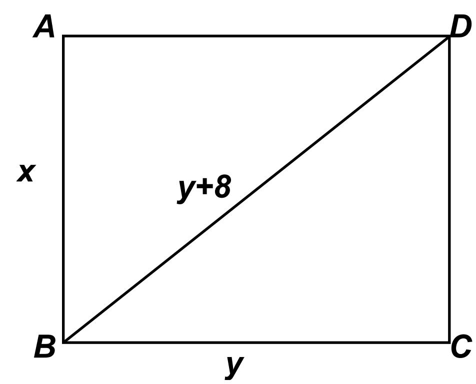
B) ______________________________

C) ______________________________

A) Compute the value of x for which $x\%$ of 80 equals 45% of 32.

B) A car travels 240 miles in a certain number of hours. If the rate of the car is increased by 18 mph, the trip will take three less hours. Compute this faster rate that will shorten the time of the trip.

C) Compute the <u>minimum</u> area of rectangle $ABCD$ if x and y are integers and the perimeter of ΔBAD exceeds 100.



MASSACHUSETTS MATHEMATICS LEAGUE
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Round 4

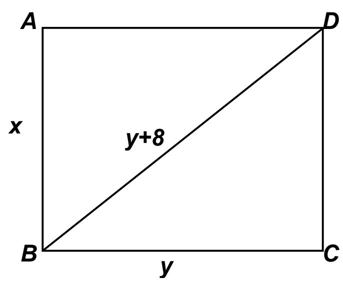
A) 45% of 32 = $\dfrac{45}{100}\cdot 32 = \dfrac{9}{20}\cdot 32 = \dfrac{9}{5}\cdot 8 = \dfrac{72}{5}$.

x% of 80 = $\dfrac{x}{100}\cdot 80 = \dfrac{4x}{5} = \dfrac{72}{5} \Rightarrow x = \underline{\mathbf{18}}$.

B) $\dfrac{240}{x}+18=\dfrac{240}{x-3} \Rightarrow 240(x-3)+18(x)(x-3)=240x \Rightarrow 18x^2-54x-720=0 \Rightarrow$

$x^2 - 3x - 40 = (x - 8)(x + 3) = 0 \Rightarrow x = 8$. Thus, the faster rate is $\dfrac{240}{8-3} = \underline{\mathbf{48}}$ mph.

C) $x^2+y^2=(y+8)^2 \Rightarrow x^2=16y+64=16(y+4) \Rightarrow \boxed{x=4\sqrt{y+4}}$



We try $y = 5, 12, 21, 32, 45, \ldots$ to make the radicand a perfect square.

$y=5 \Rightarrow (x, y+8)=(12,13) \Rightarrow \text{Per} = 5+12+13=30$

$y=12 \Rightarrow (x, y+8)=(16,20) \Rightarrow \text{Per} = 12+16+18=48$

$y=21 \Rightarrow (x, y+8)=(20,29) \Rightarrow \text{Per} = 70$

$y=32 \Rightarrow (x, y+8)=(24,40) \Rightarrow \text{Per} = 96$

$y=45 \Rightarrow (x, y+8)=(28,53) \Rightarrow \text{Per} = 126$ Bingo!

The area of $ABCD$ is $xy = 28\cdot 45 = \underline{\mathbf{1260}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019
ROUND 5 PLANE GEOMETRY: CIRCLES

ANSWERS

A) ________________________________

B) (______________ , ______________)

C) (_______ , _______ , _______)

(_______ , _______ , _______)

A) An equilateral triangle is inscribed in a circle whose <u>circumference</u> is 16π. Compute the <u>perimeter</u> of this triangle.

B) In circle O, minor arcs $\overset{\frown}{CD}=(10x-6)^\circ$, $\overset{\frown}{AB}=(4x+8)^\circ$,

$m\angle P=(2x+6)^\circ$

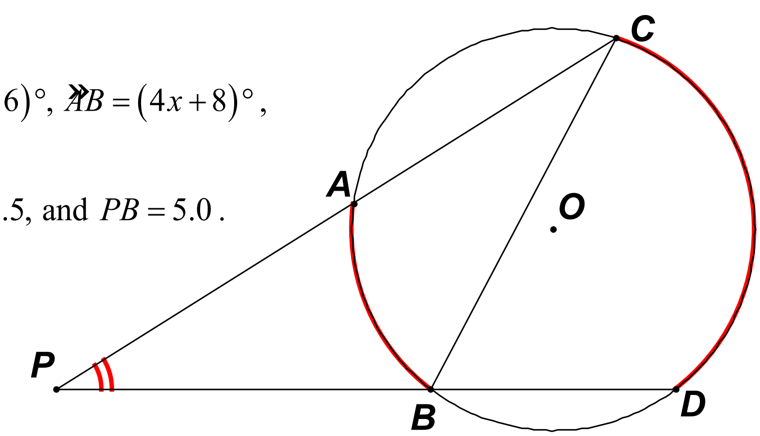
To the nearest 0.1, $PA=5.3$, $AC=3.5$, and $PB=5.0$.

<u>Diagram is drawn to scale</u>.

Let $k=m\angle CBD$ (in degrees) and

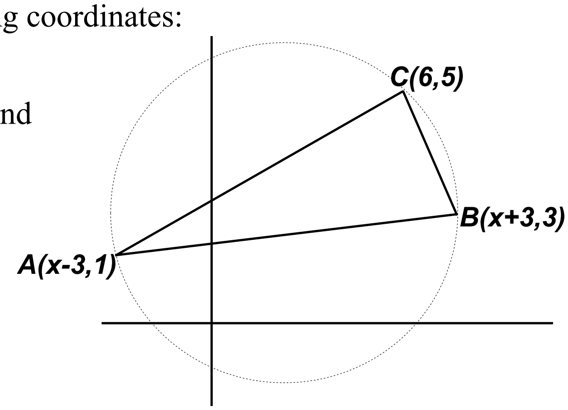
$j=BD$ (**to the nearest tenth**).

Compute the ordered pair (k,j).



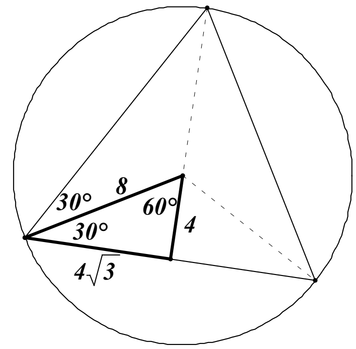
C) In ΔABC, $m\angle C=90^\circ$ and the vertices have the following coordinates: $A(x-3,1)$, $B(x+3,3)$, $C(6,5)$

The center of a circumscribed circle of ΔABC is (p,q) and its radius is r. Compute <u>both</u> ordered triples (p,q,r).



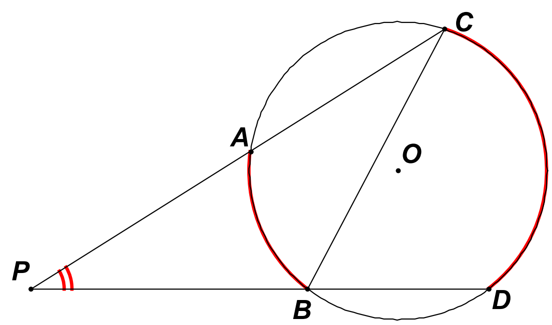
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019 SOLUTION KEY

Round 5



A) $C = 16\pi \Rightarrow r = 8$

Therefore, we have a $30-60-90$ triangle and each side of the triangle has length $2\left(4\sqrt{3}\right) \Rightarrow$ Per = $\underline{\mathbf{24\sqrt{3}}}$.



B) As an angle formed by two secant lines, we have

$$m\angle P = \frac{\overset{\frown}{CD} - \overset{\frown}{AB}}{2} = \frac{(10x-6)^\circ - (4x+8)^\circ}{2} = (2x+6)^\circ$$

$$\Leftrightarrow \frac{(6x-14)^\circ}{2} = (3x-7)^\circ = (2x+6)^\circ$$

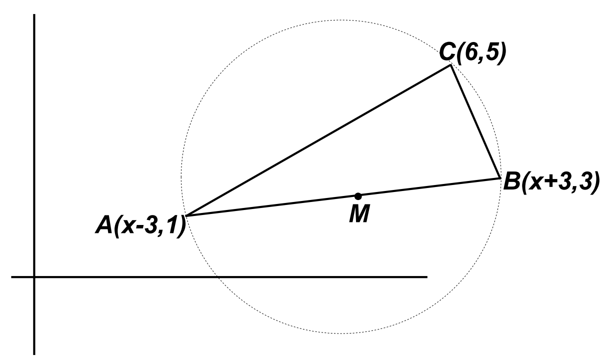
$$\Rightarrow x = 13 \Rightarrow \overset{\frown}{CD} = 10\cdot 13 - 6 = 124 \Rightarrow$$

$$k = m\angle CBD = \frac{1}{2}(124) = 62.$$

Using power of a point, we have $PA\cdot PC = PB\cdot PD \Leftrightarrow 5.3(8.8) = 5(j+5)$

$$\Rightarrow j = \frac{46.64-25}{5} = \frac{21.64}{5} = 4.328.$$

Thus, $(k, j) = \underline{\mathbf{(62,\ 4.3)}}$.



C) $A(x-3,1), B(x+3,3), C(6,5)$

$$\overline{AC} \perp \overline{BC} \Rightarrow m_{\overline{AC}} \cdot m_{\overline{BC}} = -1$$

$$\Rightarrow \left(\frac{1-5}{x-3-6}\right)\left(\frac{3-5}{x+3-6}\right) = -1$$

$$\Rightarrow \left(\frac{-4}{x-9}\right)\left(\frac{-2}{x-3}\right) = -1$$

$$\Rightarrow 8 = -x^2 + 12x - 27 \Rightarrow x^2 - 12x + 35 = (x-5)(x-7) = 0 \Rightarrow x = 5, 7$$

The center of the circumcircle is at the midpoint M of the hypotenuse, and $r = \frac{AB}{2}$

$x = 5 \Rightarrow A(2,1),\ B(8,3) \Rightarrow M(5,2)$ and $r = \sqrt{3^2+1^2} = \sqrt{10}$

$x = 7 \Rightarrow A(4,1),\ B(10,3) \Rightarrow M(7,2)$ and $r = \sqrt{3^2+1^2} = \sqrt{10}$

Thus, $x = 5 \Rightarrow (p,q,r) = \underline{\mathbf{\left(5, 2, \sqrt{10}\right),\ \left(7, 2, \sqrt{10}\right)}}$.

Also note that $AB = \sqrt{\left((x+3)-(x-3)\right)^2 + (3-1)^2} = \sqrt{40} = 2\sqrt{10} \Rightarrow r = \sqrt{10}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) A decreasing geometric progression has a first term of $12\left(2^{10}\right)$ and a common ratio of $\frac{1}{4}$.
How many terms in this progression are integers?

B) Compute $\sum_{n=1}^{n=23}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$.

Note: Σ is the add-a-bunch or summation symbol, e.g., $\sum_{n=1}^{n=4} n^2 = 1^2+2^2+3^2+4^2 = 30$.

C) In an arithmetic progression whose third term $\left(t_3\right)$ is 14, we observe that $t_4+t_7=\frac{t_{11}+t_{12}}{2}$.
Compute t_{2020}.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019 SOLUTION KEY

Round 6

A) The first term of the progression is $3 \cdot 2^{12}$ and the ratio is $\frac{1}{2^2}$ or 2^{-2}. For successive terms, the power of 2 will decrease by 2, producing exponents of 12, 10, 8, 6, 4, 2, 0.
Thus, after $3\left(2^0\right) = 3$, subsequent terms will be fractional and only **7** terms will be integers.

B) $\sum_{n=1}^{n=23}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

$\Leftrightarrow \left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right)+\ldots+\left(\cancel{\frac{1}{24}}-\frac{1}{25}\right)=\frac{1}{2}-\frac{1}{25}=\frac{25-2}{2 \cdot 25}=\underline{\mathbf{\frac{23}{50}}}$.

C) $t_3 = a + 2d = 14 \Rightarrow a = 14 - 2d$.
$t_4 + t_7 = \frac{t_{11}+t_{12}}{2} \Rightarrow 2a + 9d = \frac{2a + 21d}{2} \Rightarrow 2a = 3d$.
Substituting, $2(14-2d) = 3d \Rightarrow 7d = 28 \Rightarrow d = 4,\ a = 6$.
$t_{2020} = a + 2019d = 6 + 2019(4) = \underline{\mathbf{8082}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019
ROUND 7 TEAM QUESTIONS
ANSWERS

D) ___________________________

A)__________________________ E) (_______ , _______ , _______)

B) __________________________ F) (____________ , ___________)

C) ____

A) When $P(x)$ is divided by $x + 2$, the remainder is 62, and when $P(x)$ is divided by $x + 4$, the remainder is 68. Find the remainder when $P(x)$ is divided by $(x + 2)(x + 4)$.

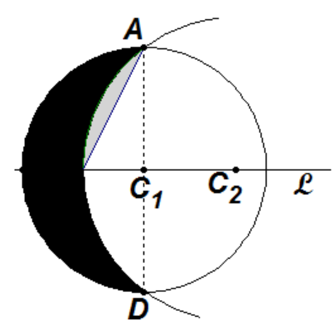
B) The first 100 natural numbers are written on a blackboard. The first pair of numbers are replaced by their sum plus their product. Now there are only 99 numbers. This process is repeated 98 more times until there is exactly one LARGE number K left. The amazing fact is that no matter what pairs of numbers are replaced by their sum plus their product, no matter if the list is shuffled in the process, after 99 iterations of these replacements, the <u>same</u> K-value is obtained. Determine a simplified expression for K.

C) Given: $0 < n < 7$, $0 \le A \le 90$, and $B = \cos\left(\text{Sin}^{-1}\left(\frac{n}{7}\right) + Cot^{-1}\left(2\sqrt{A}\right)\right)$

A and n are integers for which B is a rational number.
Compute <u>all</u> possible ordered triples (n, A, B).

D) At top speed, rented motorcycle #1 can travel 270 miles in the same time that rented motorcycle #2 can travel 240 miles. During an actual run over the Isle of Man Superbike TT course, motorcycle #1 traveled 200 miles, while motorcycle #2 traveled the same distance in 18 minutes and 45 seconds less time. Compute the top speed of rented bike #1.

E) The darkly shaded crescent is formed by circle C_2 of radius 5 overlapping circle C_1 of radius 4, where $C_1C_2 = 3$.
A, C_1, and D are collinear. The lightly shaded region bounded by the arc of C_2 and the chord of C_2 is k units. The area of the crescent can be written as $p(\pi - q) - rk$. Compute the ordered triple of integers (p, q, r).



F) Consider the following recursively defined function:

$$f(n) = \begin{cases} f(n-1) + f(n-3) & \text{for } n \ge 3 \\ f(0) = f(1) = f(2) = 1 \end{cases}.$$

A student poses the explicit formula $g(n) = \dfrac{n(n-7)}{2} + 9$ for evaluating $f(n)$ for $n \ge 4$

Unfortunately, this formula is incorrect. For some minimum value of k, $f(k) - g(k) = E > 100$. Compute the ordered pair (k, E).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019 SOLUTION KEY

Team Round

A) $P(x) = Q(x)(x + 2)(x + 4) + R(x)$, $P(-2) = R(-2) = 62$, $R(-4) = 68$
Since the divisor is 2nd degree, the remainder will be in the worse case scenario 1st degree,
i.e., $R(x) = Ax + B$. Thus, $\begin{cases} -2A + B = 62 \\ -4A + B = 68 \end{cases} \Rightarrow (A, B) = (-3, 56) \Rightarrow R(x) = \underline{\mathbf{-3x + 56}}$.

B) 100 is a lot of numbers to write on the blackboard.
Why not start with fewer and look for a pattern?

$$1\ 2 \Rightarrow 3 + 2 = 5$$

$$1\ 2\ 3 \Leftrightarrow \begin{cases} (1,2)\ 3 \Rightarrow 5\ 3 \Rightarrow 8 + 15 = 23 \\ (1,3)\ 2 \Rightarrow 7\ 2 \Rightarrow 9 + 14 = 23 \\ (2,3)\ 1 \Rightarrow 11\ 1 \Rightarrow 12 + 11 = 23 \end{cases}$$

$$(1\ 2)\ 3\ 4 \Rightarrow (5\ 3)\ 4 \Rightarrow \underline{23}\ 4 \Rightarrow 27 + 92 = 119$$

$$1\ \underline{2}\ 3\ \underline{4}\ 5 \Rightarrow \underline{14}\ \ 1\ 3\ \underline{5} \Rightarrow \underline{89}\ (1\ 3) \Rightarrow \underline{89}\ \ 7 \Rightarrow 96 + 623 = 719$$

$$1\ 2\ 3\ 4\ 5\ 6 \Rightarrow 5\ 3\ 4\ 5\ 6 \Rightarrow \underline{23}\ \ 4\ 5\ 6 \Leftrightarrow \underline{119}\ \ 5\ 6 \Rightarrow \underline{719}\ 6 \Rightarrow 725 + 4314 = 5039$$

Do you see a pattern?
How can we prove this pattern holds for all sequences of consecutive natural numbers, starting with 1, regardless of which pair of numbers is replaced by their sum plus their product?

After studying the gaps for way too long, the first pattern I noticed was this recursive relation:

N	t_n	
1	5	
2	23	
3	(23 - 5)5 + 6 + 23	119
4	(119 – 23)6 + 24 + 119	719
t_n	$(t_{n-1} - t_{n-2})(n + 2) + n! + t_{n-1}$	
	Testing…	
5	(719 – 119)7 + 120 + 719	5039

But we want an explicit expression.
Just add 1 to each of the first 5 values we have derived**:** 6, 24, 120, 720, 5040, ...
Factorial numbers! Did I ever feel like a dummy.
$n! - 1 \Rightarrow \underline{\mathbf{100! - 1}}$.
How do you prove that the relationship holds for all n?
You may want to skip the next page until you have thought about this yourself.

Team Round - continued

Here's an argument courtesy of Rick Yanco (Worcester Academy):

Key observation - Rewriting a sum plus a product: $\boxed{(a+b)+ab=(a+1)(b+1)-1}$

We'll illustrate the strategy for 5 arbitrary integers *a*, *b*, *c*, *d*, *e*.
The argument is easily extended to any number of integers.
Without any loss of generality, let's replace (*a*, *b*) and (*c*, *d*). Then:

$a,b,c,d,e \Rightarrow \underbrace{a,b}_{\left[(a+1)(b+1)-1\right]}, \underbrace{c,d}_{\left[(c+1)(d+1)-1\right]}, e$. The sequence of 5 has become a sequence of 3.

Replace these new first two terms, using the boxed expression,

$$\left[(a+1)(b+1)-1+1\right]\left[(c+1)(d+1)-1+1\right]-1=(a+1)(b+1)(c+1)(d+1)-1$$

Now we have a sequence of two terms: $(a+1)(b+1)(c+1)(d+1)-1,\ e$

One last time, after adding 1 to each factor, and subtracting 1 from the product, we have

$(a+1)(b+1)(c+1)(d+1)(e+1)-1$

If $a=1,\ b=2,\ c=3,\ d=4,\ e=5$, we have $2\cdot3\cdot4\cdot5\cdot6=6!-1$.

<u>FYE:</u> What values do each of the following sequences produce?

2, 4, 6, 8, 10

1, 3, 5, 7, 9

1, 2, 3, 5, 8, 13

3, 1, 4, 1, 5, 9, 2, 6, 5

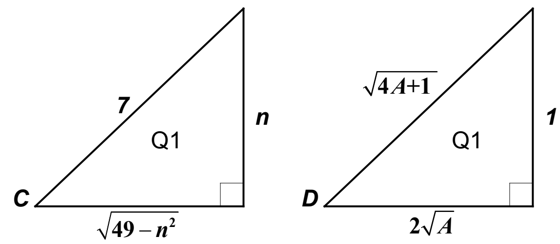
Team Round - continued

C) Given: $B = \cos\left(\operatorname{Sin}^{-1}\left(\frac{n}{7}\right) + Cot^{-1}\left(2\sqrt{A}\right)\right)$

for $0 < n < 7$ and $0 \le A \le 90$

For $(C,D) = \left(\operatorname{Sin}^{-1}\left(\frac{n}{7}\right), \cot^{-1}\left(2\sqrt{A}\right)\right)$,

using the expansion $\cos(C+D) = \cos C \cos D - \sin C \sin D$, we have



$$\cos(C+D) = \frac{\sqrt{49-n^2}}{7} \cdot \frac{2\sqrt{A}}{\sqrt{4A+1}} - \frac{n}{7} \cdot \frac{1}{\sqrt{4A+1}} = \frac{2\sqrt{49-n^2}\sqrt{A} - n}{7\sqrt{4A+1}}.$$

n	$2\sqrt{49-n^2} \cdot \sqrt{A} - n$	***Try A =***	**For *A* =**	$4A+1 =$ perfect sq.	***B***
1	$8\sqrt{3A}-1$	0,3,12,27,48,75	0, 12	1, 49	$-\frac{1}{7}, \frac{47}{49}$
2	$6\sqrt{5A}-2$	5,20,45,80	20	81	$\frac{58}{63}$
3	$4\sqrt{10A}-3$	10, 40, 90	90	$361 = 19^2$	$\frac{117}{133}$
4	$2\sqrt{33A}-4$	33	none		
5	$4\sqrt{6A}-5$	6, 24, 54	6	25	$\frac{19}{35}$
6	$2\sqrt{13A}-6$	13,52	none		

Thus, there are 5 ordered triples with *n* and *A* in the specified range for which *B* is rational, namely, $\underline{\left(\mathbf{1,0,-\frac{1}{7}}\right), \left(\mathbf{1,12,\frac{47}{49}}\right), \left(\mathbf{2,20,\frac{58}{63}}\right), \left(\mathbf{3,90,\frac{117}{133}}\right), \left(\mathbf{5,6,\frac{19}{35}}\right)}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019 SOLUTION KEY

Team Round - continued

D) From the first stated condition, the top speed of cycle #1 is $\frac{9}{8}$ of the top speed of cycle #2.

18 minutes and 45 seconds is equivalent to $\frac{18\frac{3}{4}}{60}=\frac{75}{4(60)}=\frac{5}{16}$ hour.

Let T denote the time taken to complete the 200-mile course on the faster motorcycle.

From the second condition, $\frac{200}{T}=\frac{9}{8}\left(\frac{200}{T+\frac{5}{16}}\right)$ or $\frac{8}{T}=\frac{9}{T+\frac{5}{16}}$, since factors of 200 are irrelevant.

Cross multiplying, we have $9T=8T+2.5\Rightarrow T=2.5$

Thus, the top speed of the faster bike is $\frac{200}{2.5}=\underline{\mathbf{80}}$ mph.

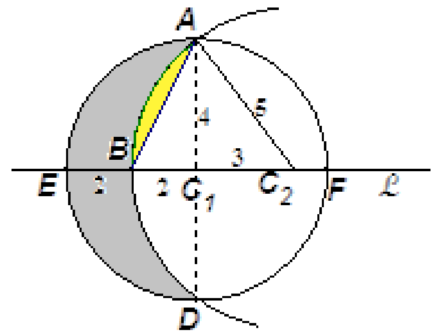
(The top speed of the slower bike is $\frac{8}{9}$ of this, or, approximately 71.1 mph.)

FYI:
On the Isle of Man, spectators can rent motorcycles to ride on the official course the week before the Superbike TT Race. These motorcycles have governors to prevent injury (or death) to inexperienced riders due to excess speed. During the actual race, the professional racers reach speeds in excess of 200 mph. James Hillier was clocked at **206 mph** on the famous Sulby straight riding a Kawasaki Ninja H2R in a *parade* lap.

Sulby Straight – Isle of Man TT Race

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019 SOLUTION KEY

Team Round - continued



E) $AC_1 = 4$

$C_2A = C_2B = 5$, $C_1C_2 = 3 \Rightarrow BC_1 = 2$

Plan: Since the crescent is divided into 2 congruent pieces by **ℒ,** and the segment on $\overline{AB}$, i.e., the lightly shaded **region**, has area *k*, we only need to add the area of the segment on $\overline{AB}$ and the area of ΔBAC_1, double the sum, and subtract this sum from the area of the semi-circle on $\overline{AD}$.

$\frac{1}{2}\left(\pi \cdot 4^2\right) - 2\left(k + \frac{1}{2}\cdot 2 \cdot 4\right) = 8\pi - 2(k+4) = 8(\pi - 1) - 2k \Rightarrow \underline{\mathbf{(8,1,2)}}$.

F) Applying the recursive definition, the first several values of $f(n)$ can be evaluated. Organizing them in a chart makes the job easy. The recursive definition states that, moving from left to right, an empty "cell" is filled with the sum of the first and third number in the group immediately preceding the empty cell. Note that the *gap* between successive terms of $g(n)$ is <u>increasing by 1</u> as we move from left to right.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
f(n)	1	1	1															
f(n)	1	1	1	**2**	**3**	**4**	**6**	**9**	**13**	**19**	**28**	**41**	**60**	**88**	**129**	**189**		
g(n)					3	4	6	9	13	18	24	31	39	48	58	69		
Diff (*E*)										1	4	10	21	40	71	120		

$\therefore\ (k, E) = \underline{\mathbf{(15,\ 120)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2019 ANSWERS

Round 1 Algebra 2: Algebraic Functions

A) -8 B) $\left(\frac{3}{4},-\frac{1}{2},2\right)$ C) $(-1, 2, -5, 0)$

Round 2 Arithmetic/ Number Theory

A) 279 B) (5, 2) C) 13

Round 3 Trig Identities and/or Inverse Functions

A) $\frac{2\pi}{3}$ B) -2 C) 2

Round 4 Algebra 1: Word Problems

A) 18 B) 48 mph C) 1260

Round 5 Geometry: Circles

A) $24\sqrt{3}$ B) $(62, 4.3)$ C) $\left(5,2,\sqrt{10}\right)$, $\left(7,2,\sqrt{10}\right)$

Round 6 Algebra 2: Sequences and Series

A) 7 B) $\frac{23}{50}$ C) 8082

Team Round

A) $-3x + 56$ D) 80

B) 101! - 1 E) (8, 1, 2)

C) $\left(1,0,-\frac{1}{7}\right),\left(1,12,\frac{47}{49}\right),\left(2,20,\frac{58}{63}\right),\left(3,90,\frac{117}{133}\right),\left(5,6,\frac{19}{35}\right)$ F) $(15,120)$

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Diagram for solution of Team E