MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

ANSWERS

A) (____________ , _____________)

B) ________________________________

C) (____________ , _____________)

A) The circle O has equation $x^2 + y^2 - 12x + 8y + 27 = 0$. O intersects the x-axis m times and the y-axis n times. Compute the ordered pair (m, n).

B) $P(x, y)$, where $x > 0$ and $y > 0$, is an endpoint of a focal chord of the ellipse defined by $9x^2 + 5y^2 = 45$. F_1 and F_2 are the foci of this ellipse. Compute $PF_1 + PF_2$.
Note: A focal chord of an ellipse (or latus rectum) is a chord that contains a focus, and that is perpendicular to the major axis of the ellipse.

C) A parabola P_1 with vertical axis of symmetry has a vertex at (0, 0) and a focus at (0, 1).
A parabola P_2 with horizontal axis of symmetry has the same focus as P_1.
If $P_1 \cap P_2 = \{(0,0), (a,b)\}$, compute the ordered pair (a, b).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019 SOLUTION KEY

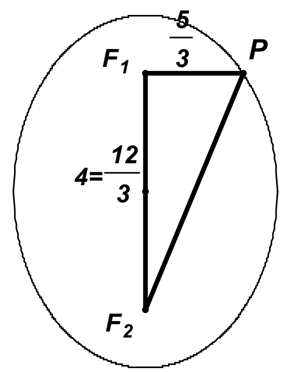
Round 1

A) $\left(x^2-12x\underline{\underline{+36}}\right)+\left(y^2+8y\underline{\underline{+16}}\right)=-27+36+16 \Leftrightarrow (x-6)^2+(y+4)^2=25$

Thus, circle O has center at $(6,-4)$ and radius 5.

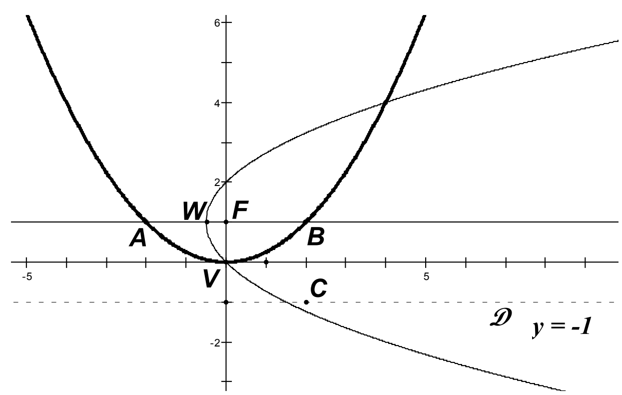
Since the shortest distance from the center of O to the y-axis is 6, there are no points of intersection with the y-axis. However, the shortest distance from the center to the x-axis is 4, so there are 2 points of intersection with the x-axis. Thus, $(m,n)=\underline{\mathbf{(2,0)}}$.

B) Since $9x^2+5y^2=45 \Leftrightarrow \frac{x^2}{5}+\frac{y^2}{9}=1$, this ellipse is vertical, has center at (0,0) with $a = 3$ and $b = \sqrt{5}$. For an ellipse, $b^2+c^2=a^2$; therefore, $c = 2$ and the foci are located at (0, ±2). The length of a focal chord is $\frac{2b^2}{a}=\frac{10}{3}$ and P has coordinates $\left(\frac{5}{3},2\right)$. $PF_1 = 5/3$. From the diagram, we see that $PF_2 = 13/3$ (based on the 5 - 12 - 13 Pythagorean Triple) and, consequently, $PF_1 + PF_2 = 18/3 = \underline{6}$.
All this is avoided, if we recall that $PF_1 + PF_2 = 2a$.
Knowing $a = 3$ gives us **6** immediately.



C) The equation of P_1 is $y=\frac{x^2}{4}$. If you need to see a derivation of this fact, it is given at the end of this solution key.

For the horizontal parabola, $VF=1 \Rightarrow a=\frac{1}{2} \Rightarrow W\left(-\frac{1}{2},1\right)$ and the equation of P_2 is $\left(x+\frac{1}{2}\right)=\frac{1}{2}(y-1)^2$.



Solving simultaneously, $2\left(x+\frac{1}{2}\right)=2x+1=(y-1)^2=\left(\frac{x^2}{4}-1\right)^2=\frac{\left(x^2-4\right)^2}{16}$

$\Rightarrow \left(x^2-4\right)^2=16(2x+1) \Leftrightarrow x^4-8x^2=32x \Leftrightarrow x\left(x^3-8x-32\right)=0$

$x^3-8x-32=0$ gives us the second point of intersection. Fortunately, the root is rational, and the rational root test tells us that the only rational roots are integers!! Testing by synthetic substitution, we quickly have $x^3-8x-32=(x-4)\left(x^2+4x+8\right)$ and the root is $x=4$. Thus, the required ordered pair is $\left(4,\frac{4^2}{4}\right)=\underline{\mathbf{(4,4)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019
ROUND 2 ALGEBRA: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) $ab = 16,016$, where a and b are positive integers of the same parity.
Compute <u>at least four</u> possible sums of a and b.

B) Given: $a^4 + b^4 = 328$, $ab\left(a^2 + b^2\right) = 120$, and $\left(ab\right)^2 = 36$, where $a > 0$ and $b > 0$.
Compute the <u>largest</u> possible value of $\left(a - b\right)^3$.

C) Given: $2r^4 - r^2 - 1 = 0$, for some real number r.
Compute <u>all</u> possible values of $r^{2020} - r^{2010} + r^{2000} - \ ... \ - r^{10} + 1$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019 SOLUTION KEY

Round 2

A) Recall: 1001 is divisible by 11, namely, $1001 = 11(91)$ and 91 = (7)(13).

$16{,}016 = 16(1001) = 2^4 \cdot 7 \cdot 11 \cdot 13$. Both *a* and *b* must contain at least one factor of 2.

There are 12 possible products.

$$16016 = \begin{cases} 2(8008) \quad 14(1144) \quad 22(728) \quad 26(616) \\ 4(4004) \quad 28(572) \quad 44(364) \quad 52(308) \\ 8(2002) \quad 56(286) \quad 88(182) \quad 104(154) \\ \cancel{16(1001)} \end{cases} \Rightarrow \begin{array}{l} \mathbf{8010,\ 4008,\ 2010} \\ \mathbf{1158,\ 750,\ 632,\ 600,\ 408,} \\ \underline{\mathbf{360,\ 342,\ 270,\ 258}} \end{array}.$$

Any four of these are required (and no incorrect answers are included).

B) Given: $a^4 + b^4 = 328$, $ab(a^2 + b^2) = 120$, and $(ab)^2 = 36$

From the 3rd equation, $ab = 6$.

Substituting $ab = +6$ in the 2nd equation, $a^2 + b^2 = 20$

Substituting for *b*, we have $a^2 + \left(\frac{6}{a}\right)^2 = 20 \Leftrightarrow a^4 - 20a^2 + 36 = (a^2 - 2)(a^2 - 18) = 0$

$\Rightarrow (a,b) = (\sqrt{2}, 3\sqrt{2})$ or $(3\sqrt{2}, \sqrt{2})$.

Thus, $(a-b)^3 = (2\sqrt{2})^3 = \underline{\mathbf{16\sqrt{2}}}$.

Checking for consistency with the 1st equation,

$a^4 + b^4 + 2(ab)^2 = (a^2 + b^2)^2 = 328 + 72 = 400 \Rightarrow (a^2 + b^2) = 20$

C) $2r^4 - r^2 - 1 = 0 \Leftrightarrow (r^2 - 1)(2r^2 + 1) = 0 \Rightarrow r = \pm 1,\ \cancel{\pm\sqrt{-\frac{1}{2}}}$

$r^{2020} - r^{2010} + r^{2000} - \ldots - r^{10} + 1$

$= (r^2)^{1010} - (r^2)^{1005} + (r^2)^{1000} - \ldots - (r^2)^5 + 1 = \underbrace{(\pm 1)^{5\cdot 202} - (\pm 1)^{5\cdot 201} + (\pm 1)^{5\cdot 200} - \ldots - (\pm 1)^{5\cdot 1}}_{202\text{ terms}} + 1$

$r = 1 \Rightarrow 101(1-1) + 1 = \underline{\mathbf{1}}$.

$r = -1 \Rightarrow 101(1+1) + 1 = \underline{\mathbf{203}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019
ROUND 3 TRIG: EQUATIONS WITH A REASONABLE NUMBER OF SOLUTIONS

ANSWERS

A) ______________________________

B) (____________ , _____________)

C) ______________________________

A) The maximum value of $\sin(3x + A°)$ occurs for $x = 25°$.

For what <u>minimum positive</u> value of x (in degrees) does $\sin(3x + A°) = \frac{1}{2}$?

B) Given: $\sin 2\theta = \tan\theta$, with θ in radians and $0 \le \theta < 2\pi$
Let N and S denote the number of solutions and the sum of these solutions, respectively.
Compute the ordered pair (N, S).

C) Given: A, B and C are angles in triangle ABC, where A is obtuse, and
$\sin A = 0.6$, $\sin B = 0.28$
Compute $\sin C$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019 SOLUTION KEY

Round 3

A) For $x = 25°$, $\sin(3x + A°) = 1 \Rightarrow 3x + A° = 90° \Rightarrow A = 90 - 75 = 15$.

$$\sin(3x+15°) = \frac{1}{2} \Rightarrow 3x+15° = \begin{cases} 30° \\ 150° \end{cases} + n(360°) \Rightarrow x = \begin{cases} 5° \\ 45° \end{cases} + n(120°)$$

Therefore, the minimum positive solution is **5**.

B) $\sin 2\theta = \tan\theta \Rightarrow 2\sin\theta\cos\theta - \dfrac{\sin\theta}{\cos\theta} = 0 \Rightarrow \dfrac{\sin\theta(2\cos^2\theta - 1)}{\cos\theta} = 0 \Rightarrow \dfrac{\sin\theta \cdot \cos 2\theta}{\cos\theta} = 0$

$\Rightarrow \cos\theta \neq 0 \left(\theta \neq \dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

$\Rightarrow \sin\theta = 0$ or $\cos 2\theta = 0$

$\Rightarrow \theta = 0, \pi$ or $2\theta = \dfrac{\pi}{2} + n\pi \Rightarrow \theta = \dfrac{\pi}{4} + \dfrac{n\pi}{2} \Rightarrow \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$.

Since none of these values are extraneous, $(N, S) = \mathbf{(6, 5\pi)}$.

C) Given: A, B and C are angles in a triangle, A is obtuse,
$\sin A = 0.6$, $\sin B = 0.28$

$A + B + C = 180° \Rightarrow C = 180° - (A + B)$

$\sin C = \sin(180 - (A+B)) = \sin(A+B) = \sin A\cos B + \sin B\cos A$

$\sin A = 0.6 = \dfrac{3}{5}$ and A obtuse $\Rightarrow \cos A = -\dfrac{4}{5}$ (using 3-4-5 right Δ)

Since A is obtuse, B must be acute and $\sin B = 0.28 = \dfrac{7}{25} \Rightarrow \cos B = \dfrac{24}{25}$ (using 7-24-25 right Δ).

Substituting, we have $\dfrac{3}{5} \cdot \dfrac{24}{25} + \dfrac{7}{25} \cdot \dfrac{-4}{5} = \dfrac{72 - 28}{125} = \mathbf{\dfrac{44}{125}}$ (or **0.352**).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019
ROUND 4 ALGEBRA 2: QUADRATIC EQUATIONS

ANSWERS

A) ______________________________

B) (____________ , _____________)

C) (________ , ________ , ________)

A) The fact that $8+64=72$ shows that the condition "the sum of a number and its square equals 72" has a positive integer solution. What is the negative integer solution?

B) The roots of $x^2+Ax+B=0$ are r_1 and r_2, where $r_2>r_1>0$. Compute the ordered pair (A,B), if the absolute value of the difference of the roots is 10 and the quotient of the roots is $\frac{8}{3}$.

C) At Fenway Park, a right-handed batter swung at a pitch 6 feet off the ground. The baseball sailed directly down the third base line and hit the base of the foul pole at the top of the "Green Monster" and the umpire indicated "homerun". The distance down the left field line from home plate to the wall is 310 feet.The height of the wall (and the base of the foul pole) is 37 feet. The ball reached a height of 213 feet directly over third base (90 feet from home plate). Assume that the arc of the ball was parabolic, i.e. satisfies an equation $y=Ax^2+Bx+C$. Compute the ordered triple (A,B,C).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019 SOLUTION KEY

Round 4

A) $x^2 + x - 72 = (x-8)(x+9) = 0 \Rightarrow x = \underline{\mathbf{-9}}$.

B) $\begin{cases} r_2 - r_1 = 10 \\ \dfrac{r_2}{r_1} = \dfrac{8}{3} \end{cases} \Rightarrow \dfrac{10 + r_1}{r_1} = \dfrac{8}{3} \Rightarrow r_1 = 6,\ r_2 = 16$.

Thus, since the sum of the roots is 22 and the product is 96, the equation is $x^2 - 22x + 96 = 0$.
$(A,B) = \underline{\mathbf{(-22,96)}}$.

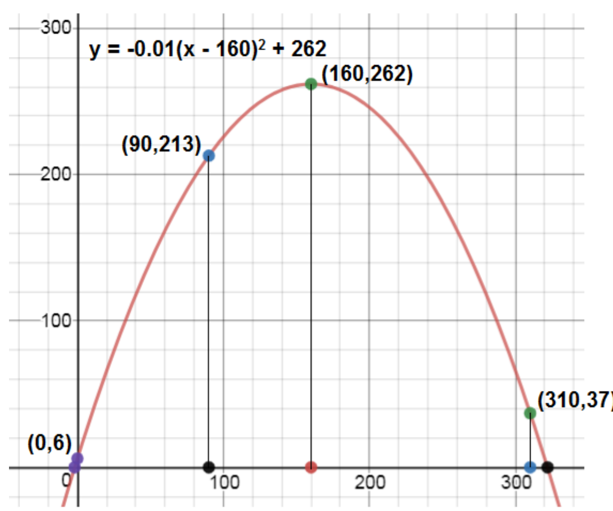
C) The quadratic equation $y = Ax^2 + Bx + C$ must be satisfied by the ordered pairs $(0,6),(90,213),(310,37)$. $(0,6) \Rightarrow C = 6$.

Substituting the other coordinates, we have $\begin{cases} 90^2 A + 90B + 6 = 213 \\ 310^2 A + 310B + 6 = 37 \end{cases}$

Subtracting 6 from both sides of each equation, we notice common factors of 9 and 31.
Dividing through by these common factors,
$\begin{cases} 900A + 10B = 23 \\ 3100A + 10B = 1 \end{cases} \Rightarrow 2200A = -22 \Rightarrow A = -0.01$.

Substituting in either equation, $B = 3.2$. Thus, $(A,B,C) = \underline{\mathbf{(-0.01,\ 3.2,\ 6)}}$.

<u>FYI</u>: By completing the square, we have $y = -0.01x^2 + 3.2x + 6 \Leftrightarrow -0.01(x-160)^2 + 262$ and the maximum height of the ball is 262 feet at a point 160 feet down the left field line from home plate. Here's the path of the ball.
Graph created at www.desmos.com.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

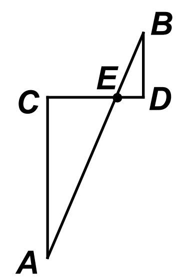
ANSWERS

A) ______________________

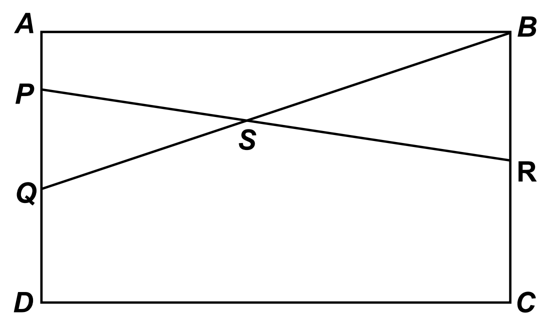
B) __________ : __________

C) ______________________

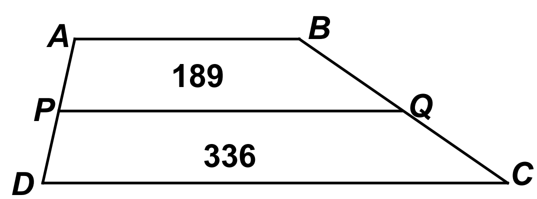
A) Given: $\overline{AC} \perp \overline{CD}$, $\overline{BD} \perp \overline{CD}$, $CE = 5$, $DE = 2$, and $AC = 12$
Compute AB.



B) In rectangle $ABCD$, $AP : PQ : QD = 4 : 5 : 6$ and $BR : RC = 10 : 11$. Compute the ratio of the area of ΔPQS to the area of ΔBRS.



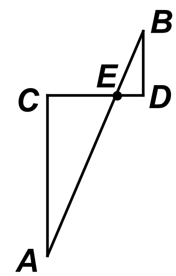
C) In trapezoid $ABCD$, $\overline{PQ} \parallel \overline{AB} \parallel \overline{CD}$, $AB = 9$.
$\overline{PQ}$ divides $ABCD$ into two similar trapezoids with the indicated areas. Let h denote the height of $PQCD$ and let k denote the length of $\overline{PQ}$.
Compute the ordered pair (h, k).



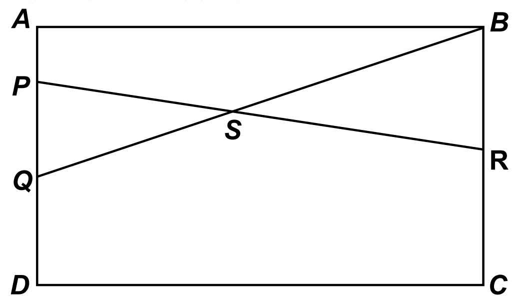
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019 SOLUTION KEY

Round 5

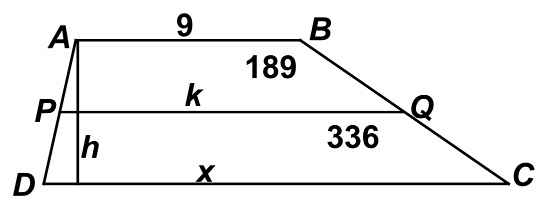
A) $AE = 13$ (Pythagorean Triple)
$\Delta ACE \sim \Delta BDE$ and the ratio of similitude i.e., the proportionality constant, is $\frac{2}{5}$. Therefore, $AB = 13 + \frac{2}{5} \cdot 13 = \frac{65+26}{5} = \mathbf{\frac{91}{5} = 18.2}$.



B) Let the common unit on $\overline{AD}$ be *a* and the common unit on $\overline{BC}$ be *b*.
$AD = BC \Rightarrow 15a = 21b \Rightarrow b = \frac{5}{7}a$
For simplicity, let $a = 7$ and $b = 5$. Then:
$AD = BC = 105$ and $(PQ, BR) = (35, 50)$
Since $\Delta PQS \sim \Delta RBS$, the required ratio is
$\left(\frac{35}{50}\right)^2 = \left(\frac{7}{10}\right)^2 = \mathbf{\frac{49}{100}}$.



C) Since $ABQP \sim PQCD$, the bases are proportional, that is $\frac{9}{k} = \frac{k}{x} \Rightarrow k^2 = 9x$. Since $ABQP \sim PQCD$, their areas are in the ratio of the square of any two corresponding sides, i.e.,
$\left(\frac{k}{x}\right)^2 = \frac{189}{336} = \frac{9}{16} \Rightarrow 16k^2 = 9x^2$. Substituting for k^2, we have $16(9x) = 9x^2$
Dividing by $9x$ (since $x \neq 0$), $x = 16 \Rightarrow k = 12$.
$\text{area}(PQCD) = \frac{1}{2}h(12+16) = 336 \Rightarrow h = \frac{336}{14} = 24$ and $(h,k) = \mathbf{(24,12)}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019
ROUND 6 ALGEBRA 1: ANYTHING

ANSWERS

A) ________________________________

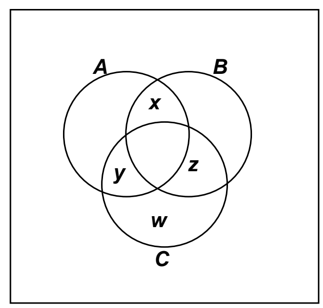
B) (______________ , ______________)

C) (______ , _____ , ______ , ______)

A) At 45 mph, I will reach my destination in 20 minutes.
At *k* mph, I would reach my destination in 8 minutes sooner. Compute *k*.

B) A line passes through the points $P(3,-2)$ and $Q(19,10)$.
Compute $R(h,k)$ on $\overleftrightarrow{PQ}$, where the *x*-coordinate exceeds the *y*-coordinate by 150.

C) Composite numbers between 1 and 60, inclusive, are placed into their homes in the Venn diagram at the right, according to their divisibility by 2 (circle *A*), 3 (circle *B*), and 5 (circle *C*).
Let *w*, *x*, *y*, and *z* denote the number of composite numbers in the indicated regions. Compute the ordered quadruple (w, x, y, z).



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019 SOLUTION KEY

Round 6

A) In 20 minutes ($\frac{1}{3}$ of an hour) my destination is 15 miles away.

$$k = \frac{15\,\text{miles}}{12\,\text{min}} = \frac{15\,\text{miles}}{1/5\,\text{hour}} = \underline{\mathbf{75}}\ \text{mph}$$

B) The slope of $\overleftrightarrow{PQ}$ is $\dfrac{10-(-2)}{19-3} = \dfrac{12}{16} = \dfrac{3}{4}$.

Thus, only points with coordinates of the form $(19+4n,\ 10+3n)$ lie on $\overleftrightarrow{PQ}$ and we have

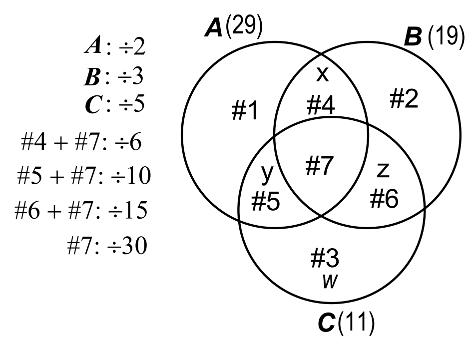
$(19+4n)-(10+3n)=150 \Rightarrow 9+n=150 \Rightarrow n=141$.

$\therefore (h,k)=(19+4\cdot 141,\ 10+3\cdot 141)=\underline{\mathbf{(583,\ 433)}}$.

Of course, using point *P* instead of *Q*,the general form of the point could have been $(3+4n,\ -2+3n)$.Show that, although *n* has a different value, point *R* has the same coordinates.

C) Excluding the primes 2, 3, and 5, there are 29 composite even numbers between 1 and 60, inclusive, 19 composite multiples of 3, and 11 composite multiples of 5. The overlap of circles *A* and *B* contains multiples of 6, i.e., 10 integers. Since region #7 (multiples of 30) contains only 2 numbers, region #4 contains 8 integers, i.e., $x = 10-2 = 8$. Similarly, $y = 6-2 = 4$, $z = 4-2 = 2$, and $w = 11-(4+2+2) = 3$.

Therefore, $(w,\ x,\ y,\ z) = \underline{\mathbf{(3,8,4,2)}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019
ROUND 7 TEAM QUESTIONS

ANSWERS

A) (_____ , _____ , _____ , ____) D) ____________________________

B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

A) $3|x-2| = 4|y+1|$ is the equation of the asymptotes of a hyperbola.
A vertex of this hyperbola is located at $V(10,k)$.
We know that eventually the graphs of the hyperbola and the asymptotes become indistinguishable, but for "small" values of *x*, there is a noticeable gap between the curve and the asymptote. At $x=18$, the distance from one of the points on the hyperbola to the closer asymptote may be written in the simplified form $\frac{a}{b}\left(c-\sqrt{d}\right)$. Compute the ordered quadruple (a,b,c,d).

B) The product $(7x-5)(5x-16)$ has the same middle term as $(15x-A)(x-B)$.
Let *N* equal the product *AB*, where *A* and *B* are both single-digit positive integers.
Factor $5(N+1)x^2 - Nx - 7(N+5)$ as the product of two binomials in *x* with integer coefficients.

C) Solve for *x* over the interval $0 \le x < \frac{\pi}{2}$: $\left(3\sin x - 4\sin^3 x\right)\left(\cos 3x\right) = 0$

D) Let r_1 and r_2 be the roots of the quadratic equation $Ax^2 + Bx + C = 0$, where *A*, *B*, and *C* are integers. Compute all possible ordered triples (A,B,C), if $\frac{1}{r_1} + \frac{1}{r_2} = -\frac{1}{10}$ and $\begin{cases} A+B+C=49 \\ BC=10 \end{cases}$.

E) A rectangular solid has integer dimensions in a ratio of $x:(2x-1):8x$ and a surface area of 760 square units. Compute the length of the space diagonal of a similar rectangular solid whose surface area is 19000 square units.
Note: A space diagonal passes through the center of the rectangular solid.

F) Determine the number of ordered pairs (A,B) for which the base 7 number $3AB4_7$ is divisible by 4.

MASSACHUSETTS MATHEMATICS LEAGUE
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Team Round

A) $3|x-2| = 4|y+1| \Leftrightarrow$

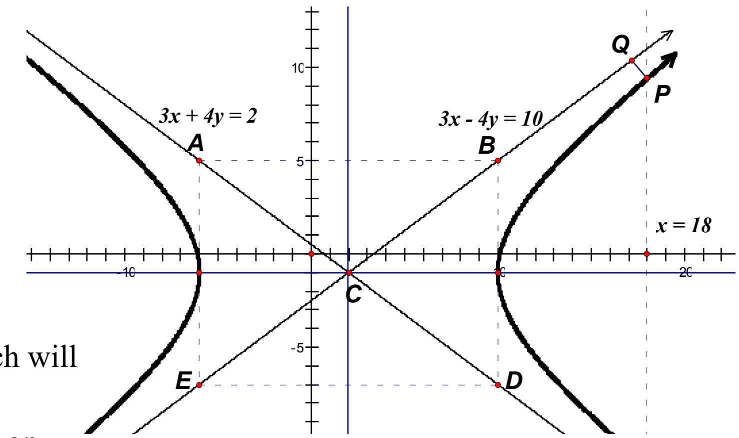
$3\sqrt{(x-2)^2} = 4\sqrt{(y+1)^2}$

Squaring both sides, $9(x-2)^2 = 16(y+1)^2$

$\Leftrightarrow (y+1)^2 = \frac{9}{16}(x-2)^2$

$\Rightarrow (y+1) = \pm\frac{3}{4}(x-2)$

Clearly, these lines intersect at $(2,-1)$ which will be the center of the hyperbola.



Since the vertices of this hyperbola must lie on either a vertical or horizontal line through the center, either the *x*-coordinate or the *y*-coordinate of point *V* must match the corresponding coordinate of the center. Therefore, $k=-1$ and the hyperbola is horizontal, and the equation must be of the form $\frac{(x-2)^2}{a^2} - \frac{(y+1)^2}{b^2} = 1$.

$a = 10-2 = 8$ and the slope of the asymptote $\overleftrightarrow{BE}$ is $\frac{b}{a} = \frac{3}{4} \Rightarrow b = 6$.

At point *P*, $x = 18$ and, substituting,

$\frac{(18-2)^2}{64} - \frac{(y+1)^2}{36} = 1 \Rightarrow 4 - \frac{(y+1)^2}{36} = 1 \Rightarrow (y+1)^2 = 36\cdot 3 \Rightarrow y = -1 \pm 6\sqrt{3}$

At *P*, $y = -1+6\sqrt{3}$. Using the point-to-line distance formula,

$PQ = \frac{\left|3\cdot 18 - 4\left(-1+6\sqrt{3}\right) - 10\right|}{\sqrt{3^2+4^2}} = \frac{48-24\sqrt{3}}{5} = \frac{24}{5}\left(2-\sqrt{3}\right)$, which is slightly more than 1.25.

$\Rightarrow (a,b,c,d) = \underline{\mathbf{(24,5,2,3)}}$.

B) The middle term in the product $(7x-5)(5x-16)$ is $-137x$. Thus, $A + 15B = 137$

Since *A* and *B* are both single digits, the only possibility is $(A, B) = (2, 9) \Rightarrow N = 18$

Therefore, we must factor $95x^2 - 18x + 161$. Since the middle coefficient is <u>even</u>, and the other coefficients are both <u>odd</u>, all the coefficients in both binomial factors must be <u>odd</u>.

Since 95 = 5(19) and 161 = 7(23), we try **<u>(19*x* + 23)(5*x* – 7)</u>** and it checks.

[Using F<u>OI</u>L, the coefficient of the middle terms is $5(23) - 7(19) = 115 - 133 = -18$.]

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019 SOLUTION KEY

Team Round - continued

C) Since $\left(3\sin x - 4\sin^3 x\right) = \sin 3x$, we have $\left(3\sin x - 4\sin^3 x\right)\left(\cos 3x\right) = 0 \Leftrightarrow \sin 3x \cos 3x = 0$

Doubling each side, and applying the double angle formula, $\sin 6x = 0$.

$\Leftrightarrow 6x = 0, \pi, 2\pi, 3\pi, \ldots \Rightarrow x = \underline{\mathbf{0, \frac{\pi}{6}, \frac{\pi}{3}}}$.

D) $\frac{1}{r_1} + \frac{1}{r_2} = -\frac{1}{10} \Leftrightarrow \frac{r_1 + r_2}{r_1 r_2} = \frac{-1}{10} \Leftrightarrow \frac{-B/A}{C/A} = \frac{-B}{C} \Rightarrow C = 10B$

Combined with $BC = 10$, we have $10B^2 = 10 \Rightarrow B = \pm 1$

$B = 1 \Rightarrow (A,B,C) = \underline{\mathbf{(38,1,10)}}$. $B = -1 \Rightarrow (A,B,C) = \underline{\mathbf{(60,-1,-10)}}$.

E) Since $\frac{19000}{760} = \frac{19(1000)}{19(40)} = 25$, the ratio of the edges of the two similar rectangular solids is 5 : 1.

The surface area of an L x W x H rectangular solid is $2(LW + LH + WH)$

$= 2\left(x(2x-1) + x(8x) + (2x-1)(8x)\right) = 2\left(2x^2 - x + 8x^2 + 16x^2 - 8x\right) \Rightarrow 2\left(26x^2 - 9x\right) = 760$

$\Rightarrow 26x^2 - 9x - 380 = 0$

The odd middle term limits the options to be tested, since the factors must be of the form $\left(x - \boxed{even}\right)\left(26x - \boxed{odd}\right)$ and the only even factors of 380 are 2, 4, and 380.

~~$(x-2)(26x+190)$~~, $(x-4)(26x+95) \Rightarrow 95 \cdot 1 - 4 \cdot 26 = 95 - 104 = -9$ Bingo!

The edges of the original solid are 4, 7, and 32.

The space diagonal is $\sqrt{4^2 + 7^2 + 32^2} = \sqrt{16 + 49 + 1024} = \sqrt{1089} = 33$

The space diagonals are linear dimensions, so their lengths must be in the same ratio as corresponding edges. Therefore, the new space diagonal has length **<u>165</u>**.

Alternately, try $x = 1, 2, 3, 4, \ldots$. The 4th try hits pay dirt.

F) $3AB4_7 = 7^3 \cdot 3 + 7^2 \cdot A + 7 \cdot B + 4 = 1029 + 49A + 7B + 4 = 4k$

Thus, $\frac{1033 + 49A + 7B}{4}$ must be an integer.

$\frac{1033 + 49A + 7B}{4} = \frac{1032 + 48A + 4B}{4} + \frac{1 + A + 3B}{4} = 258 + 12A + B + \frac{1 + A + 3B}{4}$

$1 + A + 3B$ must be a multiple of 4. A chart will help keep track of the possibilities.

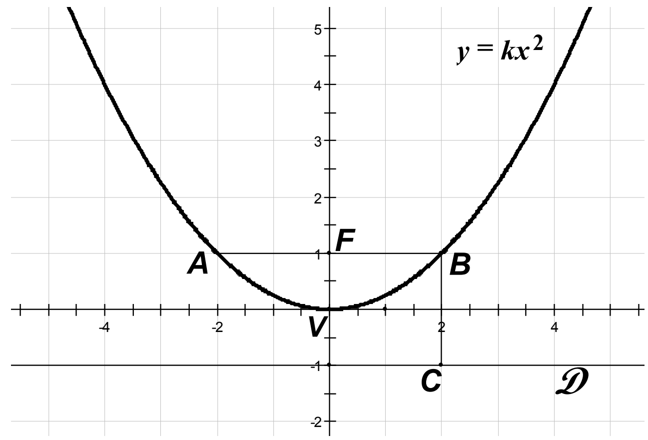
Note: A 0-digit is allowed.

1+A+3B	A+3B	(A, B)	1+A+3B	A+3B	(A, B)
0	-1	None	16	15	(6,3), (3,4), (0,5)
4	3	(3,0), (0,1)	20	19	(4,5), (1,6)
8	7	(4,1), (1,2)	24	23	(5,6)
12	11	(5,2), (2,3)			

Thus, there are **<u>12</u>** possible ordered pairs.

Here's the reasoning for 1C:

We know the equation of P_1 must be of the form $y = kx^2$. $\overline{AB}$, the focal width, is the segment perpendicular to the axis of symmetry passing through the focus $F(0, 1)$. Since a parabola is the set of points equidistant from a fixed point (the focus) and a fixed line $\mathcal{D}$(called the directrix) and the vertex $V(0, 0)$ is a point on the parabola, the fixed line must be $y = -1$. Let $\overline{BC} \perp \mathcal{D}$. Since B is a point on the parabola, $BF = BC = 2$, $B(2,1)$, and the focal width $AB = 4$. Substituting, $1 = k(2)^2 \Rightarrow k = \frac{1}{4}$.



In general, $(y-k) = \frac{1}{4|a|}(x-h)^2$ is the equation of a vertical parabola with vertex at (h,k) and focus a units from the vertex.

For $a > 0$, the parabola opens up; for $a < 0$, it opens down.

Similarly, $(x-h) = \frac{1}{4|a|}(y-k)^2$ is the equation of a horizontal parabola.

For $a < 0$, the parabola opens to the left; $a > 0$, it opens to the right.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2019 ANSWERS

Round 1 Analytic Geometry: Anything

A) (2, 0) B) 6 C) (4, 4)

Round 2 Algebra: Factoring

A) 2010, 4008, 8010
1158, 750, 632, 600,
408, 360, 342, 270, 258
(Any 4 of these 12 numbers.)

B) $16\sqrt{2}$

C) 1 and 203

Round 3 Trigonometry: Equations

A) 5 B) $(6, 5\pi)$ C) $\frac{44}{125}$ (or 0.352)

Round 4 Algebra 2: Quadratic Equations

A) -9 B) $(-22, 96)$ C) $(-0.01, 3.2, 6)$

Round 5 Geometry: Similarity

A) 18.2 or $\frac{91}{5}$ B) 49: 100 C) (24, 12)

Round 6 Algebra 1: Anything

A) 75 B) (583, 433) C) $(3, 8, 4, 2)$

Team Round

A) $(24, 5, 2, 3)$

B) $(19x+23)(5x-7)$

C) $0, \frac{\pi}{6}, \frac{\pi}{3}$

D) $(60, -1, -10), (38, 1, 10)$
- both answers are required

E) 165

F) 12