MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

ANSWERS

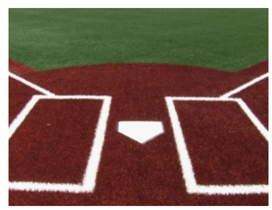
A) ______________________________

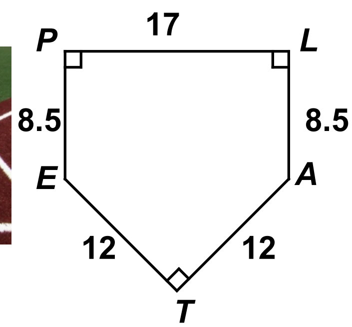
B) (____________ , ____________)

C) ______________________________

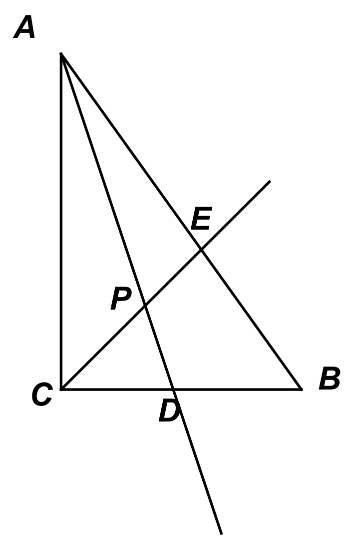
A) The hypotenuse $\overline{AB}$ of right triangle ABC has length 6, and the legs have lengths in a ratio of 3 : 4.If A is the smaller acute angle in ΔABC, compute $\sin A + \cos B$.

B) According to Major League Baseball, home plate is a pentagon with two parallel sides, 3 right angles and dimensions as indicated. Unfortunately, this is *impossible* mathematically. The angle at T <u>cannot</u> be a right angle. If $x = \cos\angle ETA$ and y is either A (for acute) or O (for obtuse), compute the ordered pair (x, y).





C) In ΔABC, $AB = 4$, $m\angle A = 30°$, $m\angle B = 60°$, $\overrightarrow{AD}$ and $\overrightarrow{CE}$ are angle bisectors and $\overrightarrow{AD} \cap \overrightarrow{CE} = \{P\}$ Compute CE.



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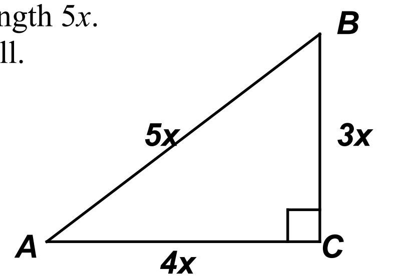
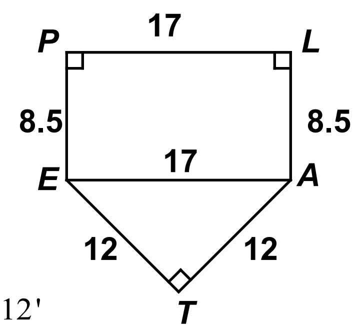
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018 SOLUTION KEY

Round 1

A) If the legs have lengths $3x$ and $4x$, then the hypotenuse must have length $5x$. Therefore, $5x = 6$ and $x = 1.2$, but we need not have solved for x at all. A is the smallest angle, since it is opposite the shortest side.
From the diagram, it is clear that $\sin A = \cos B = \frac{3x}{5x} = \frac{3}{5}$, regardless of the value of x, and the required sum is $\underline{\boldsymbol{\frac{6}{5}}}$.



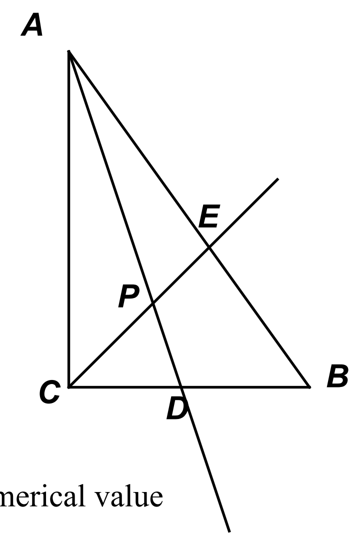
B) Draw $\overline{EA}$. PLATE "supposedly" consists of a rectangle and an isosceles right triangle. Using the Law of Cosines on ΔETA,
$17^2 = 12^2 + 12^2 - 2 \cdot 12 \cdot 12 \cdot \cos \angle ETA$
$\Rightarrow 289 = 288 - 288 \cos \angle ETA \Rightarrow \cos \angle ETA = -\frac{1}{288}$
Since the cosine is negative, $\angle ETA$ must be obtuse $\Rightarrow \underline{\boldsymbol{\left(-\frac{1}{288}, O\right)}}$.
FYI: The angle at T is approximately $90°11'56.2"$. An error of almost $12'$ (or $\frac{1}{5}$ of a degree) is an error of approximately 1 part in 450. I suppose we should cut MLB a little slack. What *percent* error is this?



C) $BC = 2, AC = 2\sqrt{3},\ m\angle ACP = 45°, m\angle CAP = 15°$
$\Rightarrow m\angle APC = 120° \Rightarrow m\angle AEC = 105°$
Using the Law of Sines,

$$\frac{CE}{\sin 30°} = \frac{AC}{\sin 105°} \Rightarrow CE = \frac{\frac{1}{2}\left(2\sqrt{3}\right)}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}} \cdot \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \underline{\boldsymbol{3\sqrt{2}-\sqrt{6}}}$$

Unfamiliar with 105° as a special angle?
Recall the sine of an angle always equals the sine of its supplement, i.e.
$\sin\theta = \sin\left(180° - \theta\right) \Rightarrow \sin 105° = \sin 75°$.
Refer to **Contest #2 November 2008** for an explanation of how this numerical value is determined without knowing fancy formulas for $\sin\left(A \pm B\right)$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018
ROUND 2 ARITHMETIC/NUMBER THEORY

ANSWERS

A) ______________________________

B) (________ , ________ , ________)

C) ______________________________

A) In the puzzle below, each card hides a base 10 digit.
What digit(s) could be hidden under the card with the X, if the 4-digit number on the right side of the equation is divisible by 3?

$$996 + \square = \square\square\square\boxed{X}$$

B) It is a hot day at the beach, and Bud and Lou have come unprepared.
They each dig into their pockets, and find some coins to buy some water.
Together they have \$2 in change.
Using only his money, Bud is \$1 short of the cost of a bottle of water.
Using only his money, Lou is \$1.50 short.
Let b, l and c denote the amount of money Bud brought, the amount of money Lou brought, and the cost of a bottle of water, respectively. If all amounts are in cents, determine the ordered triple (b, l, c).

C) Quinn wrote some base 10 digits on the chalkboard. All the digits were different. After he erased three of them, the remaining digits had a sum of 40. Considering all possible erasures, there are <u>two</u> possible largest integer multiples of 8 which can be formed using the digits that were not erased. Compute both of them.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018 SOLUTION KEY

Round 2

A) The single digit on the left side must be at least 4. $996 + \square = \square\square\square\boxed{X}$
However, the resulting sum 1000 is not divisible by 3.
$6 \Rightarrow 1002 = 3(334) \Rightarrow X = \underline{\mathbf{2}}$.
Since successive multiple of 3 are 3 apart, $9 \Rightarrow 1005 \Rightarrow x = \underline{\mathbf{5}}$.

B) $\begin{cases} b + 100 = c \\ l + 150 = c \end{cases} \Rightarrow b - l = 50$
But $b + l = 200$. Adding, $2b = 250 \Rightarrow b = 125$, $l = 75$ and $c = 225$
Therefore, (b, l, c) = **<u>(125, 75, 225)</u>**

C) Since the base 10 digits 0, 1, 2, …, 9 sum to 45, the three erased digits sum to 5.
They must be $\{0,1,4\}$ or $\{0,2,3\}$
Case 1: Available digits $\{2,3,5,6,7,8,9\}$
The 3-digit number formed by the rightmost three digits must be divisible by 8.
We are trying to form the largest multiple of 8 using 2, 3 and 5, saving the remaining digits which will be arranged from left to right in decreasing order.
Thus, we have 9876 | 352 $\Rightarrow$**<u>9876352</u>**.

Case 2: Available digits $\{1,4,5,6,7,8,9\}$
Since no multiple of 8 can be formed using 1, 4 and 5, we try 1, 4, and 6.
The only multiple of 8 is 416$\Rightarrow$**<u>9875416</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

ANSWERS

A) (________ , ________ , ________)

B) C(____ , _____) Area: _________

C) ________________________________

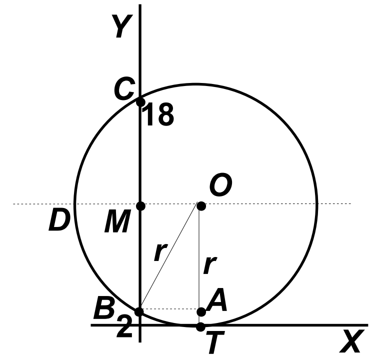
A) Circle O is tangent to the positive x-axis, and has y-intercepts at $B(0, 2)$ and $C(0,18)$.
Its equation may be expressed in the form $(x-h)^2+(y-k)^2=r^2$.
Compute the ordered triple (h,k,r).

B) Circles A and B are concentric. The radius of circle A is 60% of the radius of circle B.
The equation of circle B is $5x^2+5y^2-15y=2$.
Find the center of the circles, and the area of the region between them.

C) A circle with radius $\sqrt{20}$ is tangent to the graph of $x^2+y^2=5$ at the point $(1,-2)$. Compute <u>all</u> possible coordinates (h,k) of the center of this circle.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018 SOLUTION KEY

Round 3



A) The center of circle O must lie on the horizontal line through the midpoint M of $\overline{BC}$, specifically, on $y = 10$. Thus, the center O is located at (h, 10). Since the circle is tangent to the x-axis at point T, its radius must also be 10. In right ΔABO, $OB = r = 10,\ OA = 10 - 2 = 8 \Rightarrow AB = MO = h = 6$.

Thus, $(h,k,r) = \underline{(\mathbf{6,10,10})}$.

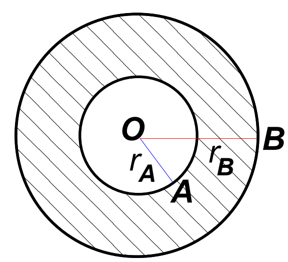
B) Completing the square, $5x^2 + 5y^2 - 15y = 2 \Leftrightarrow 5x^2 + 5\left(y - \frac{3}{2}\right)^2 = \frac{53}{4} \Leftrightarrow x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{53}{20}$

Thus, the center is $\underline{\left(\mathbf{0,\frac{3}{2}}\right)}$ and $r_B^{\ 2} = \frac{53}{20}$.

$r_A = 0.6r_B = \frac{3}{5}r_B \Rightarrow r_A^{\ 2} = \frac{9}{25}r_B^{\ 2}$. Therefore, $r_A^{\ 2} = \frac{9}{25} \cdot \frac{53}{20} = \frac{477}{500}$.

The area between the circles is

$\pi r_B^{\ 2} - \pi r_A^{\ 2} = \pi\left(r_B^{\ 2} - r_A^{\ 2}\right) = \pi\left(\frac{53}{20} - \frac{477}{500}\right) = \pi\left(\frac{1325 - 477}{500}\right) = \frac{848\pi}{500} = \underline{\mathbf{\frac{212\pi}{125}}}$.



Note that it was not necessary to solve for either r_A or r_B.

C) The center must lie on the line through (0, 0) and $(1, -2)$.

$y = -2x \Rightarrow k = -2h$

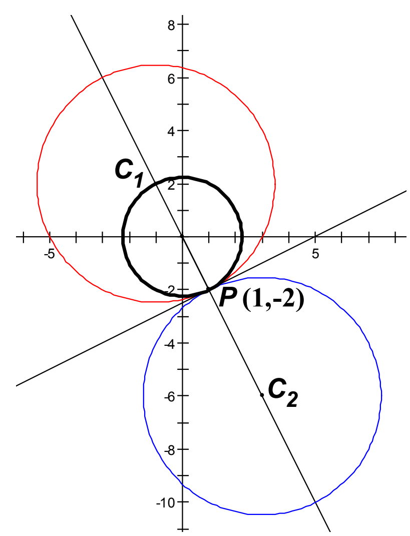
$(h-1)^2 + (k+2)^2 = 20 \Leftrightarrow h^2 - 2h + 1 + k^2 + 4k + 4 = 20$

$\Rightarrow h^2 - 2h + 1 + 4h^2 - 8h + 4 = 20$

$\Rightarrow 5h^2 - 10h - 15 = 5\left(h^2 - 2h - 3\right) = 5(h-3)(h+1) = 0$

$\Rightarrow h = 3, -1$.

Therefore, the possible centers are $\underline{(\mathbf{3,-6})}$, $\underline{(\mathbf{-1,2})}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) ______________________________

B) (____________ , ____________)

C) ______________________________

A) Solve for x over the reals. $$\log_5 3125 - 3\log_7 x + 2\log_7 x = \frac{\log_5 x}{\log_5 7} + \log_7 2401$$

If necessary, express your answer as a simplified radical.

B) The functions $f(x) = 16^{2x+1}$ and $g(x) = 8^{3x-k}$ intersect at $y = 32768$.
If the x-coordinate of the point of intersection is h, compute the ordered pair (h,k)

C) Solve for x: $$8^{12x} - 8^{9x+1} + 3\cdot 8^{6x+1} - 4\cdot 8^{3x+1} + 16 = 0$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018 SOLUTION KEY

Round 4

A) $5-3\log_7 x+2\log_7 x=\log_7 x+4$

$\Rightarrow 2\log_7 x=1\Rightarrow \log_7 x=\dfrac{1}{2}\Rightarrow x=7^{\frac{1}{2}}=\underline{\sqrt{7}}$.

B) $32768=4(8192)=16(2048)=64(512)=2^6 2^9=2^{15}$.

$16^{2h+1}=\left(2^4\right)^{2h+1}=2^{8h+4}=2^{15}\Rightarrow h=\dfrac{11}{8}$

$8^{3x-k}=\left(2^3\right)^{3x-k}=2^{9x-3k}\Rightarrow 9\left(\dfrac{11}{8}\right)-3k=15\Rightarrow k=\dfrac{99-120}{8}\cdot\dfrac{1}{3}=-\dfrac{7}{8}$

Thus, $(h,k)=\underline{\left(\mathbf{\dfrac{11}{8},-\dfrac{7}{8}}\right)}$.

<u>**FYI**</u>: In computer lingo, 1 kilobyte (K) was 1000 bytes (actually, $1024 = 2^{10}$ bytes); so, 32K (32768) was a frequently referenced number and commonly recognized as 2^{15}. In these days of gigabytes and terabytes, this is ancient history.

C) $8^{12x}-8^{9x+1}+3\cdot 8^{6x+1}-4\cdot 8^{3x+1}+16=0\Leftrightarrow\left(8^{3x}\right)^4-4(2)^1\left(8^{3x}\right)^3+6(2)^2\left(8^{3x}\right)^2-4(2)^3\left(8^{3x}\right)^1+(2)^4=0$

Notice the coefficients are from the 4th row of Pascal's Triangle, and this expansion is simply $\left(8^{3x}-2\right)^4=0\Rightarrow 8^{3x}-2=0\Rightarrow 2^{9x}=2^1\Rightarrow x=\underline{\mathbf{\dfrac{1}{9}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Given: $a : b = 7 : 10$, $b : c = 4 : 5$

Compute $\dfrac{a+c}{b+c}$.

B) Suppose in some alternate universe, the energy E (measured in joules) created by the annihilation of *m* grams of matter is given by the formula $E = kmc^2$, where the constant $c = 3\left(10^8\right)$ m/sec.

$E = 8\left(10^{16}\right)$ joules when $m = 0.8$ grams

Compute the mass *m* (in grams) which when annihilated releases 1 million joules of energy.

<u>FYI</u>: One million joules of energy is approximately the energy required to keep a human heart beating (at 80 bpm) for a day.

C) Many works of art incorporate dimensions satisfying the proportion $\dfrac{L}{W} = \dfrac{L+W}{L}$.

Suppose such a painting has a perimeter of 4 feet 4 inches.
Compute *W*, the width of the painting (in inches).


Day and Night -Woodcut by
M.C. Escher (1938)
(67.7 cm x 39.1 cm)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018 SOLUTION KEY

Round 5

A) $\begin{cases} a:b=7:10=14:\underline{20} \\ b:c=4:5=\underline{20}:25 \end{cases}$

Therefore, $a:b:c=14:20:25$, so let $a=14n,\ b=20n,\ c=25n$.

Substituting, $\dfrac{a+c}{b+c}=\dfrac{14+25}{20+25}=\dfrac{39}{45}=\underline{\dfrac{\mathbf{13}}{\mathbf{15}}}$.

B) $8\left(10^{16}\right)=k\cdot 0.8\cdot\left(3\cdot 10^{8}\right)^{2}\Rightarrow k=\dfrac{8\left(10^{16}\right)}{0.8\cdot 9\left(10^{16}\right)}=\dfrac{8}{7.2}=\dfrac{10}{9}$

Thus, we require $10^{6}=\dfrac{10}{9}m\left(9\cdot 10^{16}\right)=m\left(10^{17}\right)\Rightarrow m=\underline{\mathbf{10^{-11}}}$ grams.

C) $\dfrac{L}{W}=\dfrac{L+W}{L}\Leftrightarrow L^{2}-LW-W^{2}=0$. Dividing by W^{2},

$\left(\dfrac{L}{W}\right)^{2}-\dfrac{L}{W}-1=0$. Using the Quadratic Formula, solve for $\dfrac{L}{W}$.

$\dfrac{L}{W}=\dfrac{1\pm\sqrt{1+4}}{2}\Rightarrow\dfrac{L}{W}=\dfrac{\sqrt{5}+1}{2}$

$Per=52\Rightarrow L=26-W$

Thus, $\dfrac{26-W}{W}=\dfrac{\sqrt{5}+1}{2}\Rightarrow 52-2W=\left(\sqrt{5}+1\right)W\Rightarrow W=\dfrac{52}{\sqrt{5}+3}=\dfrac{52\left(\sqrt{5}-3\right)}{-4}=\underline{\mathbf{13\left(3-\sqrt{5}\right)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

ANSWERS

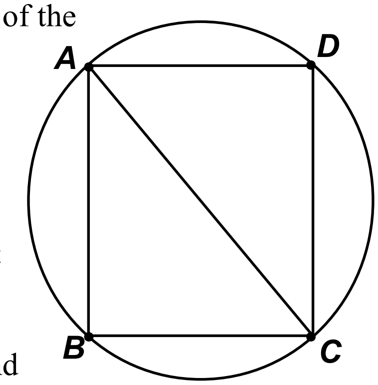
A) ______________________________

B) (___________ , ____________)

C) ______________________________

A) 16 points are located on the sides of a 5 x 5 square so that each side is divided into 5 segments of unit length. P and Q are distinct points chosen from this set of 16 points. $\overline{PQ}$ crosses the interior of the 5 x 5 square. How many distinct segments $\overline{PQ}$ have integer length.

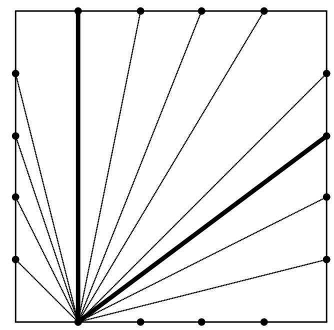
B) The diagram at the right is our practice cross-country course. A leg of the course is any one of the 4 arcs on the circle, the 4 sides of the inscribed rectangle, and the given diagonal. $AB = 8$ units and $BC = 6$ units. Our cross-country coach gave us these directions: Start at any one of the named points. Run a complete leg of the course, e.g., $\overline{AB}, \overline{BA}, \overset{\frown}{AD}, \overset{\frown}{DA}, \overline{AC}, \overline{CA}$, etc.
After completing a leg, you may not reverse direction and run over the same leg of the course. You must continue running if you are at the beginning of a leg over which you have not yet run; otherwise, your workout is over.
Let m be the length (in units) of the shortest possible practice run and M, the length (in units) of the longest possible practice run.
Compute the ordered pair (m, M).
Phew! We're very glad the coach's "unit" is only 500 yards.



C) All the diagonals in the 12-gon $V_1V_2 \ldots V_{12}$ have been drawn.
Erase the diagonal V_mV_n, if $m+n$ is divisible by 3.
How many diagonals are left, i.e., not erased?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018 SOLUTION KEY

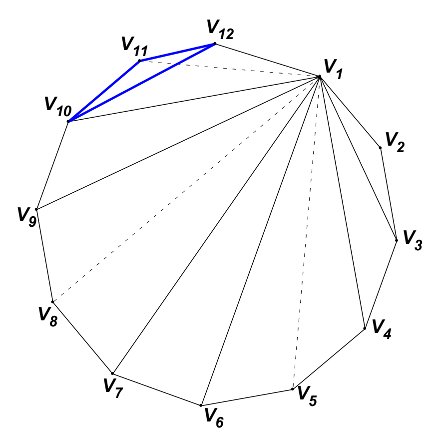
Round 6



A) For every point P, there are exactly 2 points for which PQ will be an integer, one directly opposite P, and one on an adjacent side forming a 3-4-5 right triangle.
Thus, there are $16 \cdot 2 = 32$ segments of integer length, but each segment has been counted twice, once as $\overline{PQ}$ and once as $\overline{QP}$.
Thus, there are only **16** distinct segments.

B) The diagonal of the rectangle is 10 and the perimeter of the rectangle is 28
The radius of the circle is 5, resulting is a circumference of 10π .
Starting at point A, you can complete all 9 legs of the course and will finish at point C.
Starting at C, you would end at A.
The length of the longest run is $M = 10 + 28 + 10\pi = 38 + 10\pi$.
Starting at B, the shortest path is $\overline{BC} \to \underline{CB} \to \overline{BA} \to \underline{AB}$ which is equivalent to the sum of the semi-perimeters of the rectangle and the circle $\Rightarrow m = 14 + 5\pi$. Starting at D, the shortest distance is the same. Thus, $(m, M) = \underline{\mathbf{(14 + 5\pi,\ 38 + 10\pi)}}$.

C) Pick each vertex V_m (for m = 1 through 9), pick all *subsequent* vertices V_n (for $n>m + 1$), for which the sum of the subscripts is a multiple of 3.
Why do we stop at $m = 9$? Notice, for $V_{10}V_{11}$, $m + n = 21 = 3 \cdot 7$, but $V_{10}V_{11}$ is a side not a diagonal. Also, for any *subsequent* vertex V_n, $10 + n$ will not a multiple of 3.
For m = 1, ..., 9, diagonal V_mV_n must be erased for each of the following ordered pairs:
$(m,n) = (1,5), (1,8), (1,11), (2,4), (2,7), (2,10), (3,6), (3,9), (3,12),$
$(4,8), (4,11), (5,7), (5,10), (6,9), (6,12), (7,11), (8,10), (9,12)$
A total of 18 diagonals. The total number of diagonals is
$\frac{n(n-3)}{2} = \frac{12 \cdot 9}{2} = 54$.
Thus, the number of remaining diagonals is 54 – 18 = **36**.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018
ROUND 7 TEAM QUESTIONS

ANSWERS

A)___________________________ D) ___________________________

B) (______ , ______ , ______) E) ___________________________

C) ___________________________ F) ___________________________

A) Start with a 3-4-5 right triangle. Construct a sequence of right triangles, each with integer sides, which have a short leg with the same length as the hypotenuse of its predecessor. In each triangle, the lengths of the hypotenuse and the longer leg differ by 1. Compute the perimeter of the 4th triangle in this sequence.

B) Alec makes up his own sets of rules. Here is an example from his younger days when he was in grade school $\frac{4}{1}\cdot\frac{5}{8}=\frac{45}{18}=\frac{5\cancel{(9)}}{2\cancel{(9)}}=\frac{5}{2}$. Amazingly, he's got the right answer! However, Mr. Smart-Alec's method fails more than it works. When does it work? Assuming $b>1$, $\frac{1}{b}\cdot\frac{c}{d}=\frac{(1c)_{10}}{(bd)_{10}}$ for exactly k triples (b,c,d).Let m and M denote the minimum and maximum values of $b\cdot c\cdot d$, respectively. Compute the ordered triple (k,m,M).

C) Given: $\begin{cases}3x-2y=17\\2x+3y=7\end{cases}$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the bisectors of the vertical angles formed by the given lines.
Compute the area of the triangle formed by $\mathcal{L}_1$, $\mathcal{L}_2$ and the y-axis.

D) Let $f(x)=\log_8\left(4^{x^2}\cdot16^x\cdot64^k\right)$.

For a *maximumnegative* integer k-value, there exist exactly two rational values of x for which $f(x)=4$. Call them x_1 and x_2, where $x_1>x_2$. Compute the ordered triple (k,x_1,x_2).

E) Consider fractions $\frac{P}{Q}$, for positive integers P and Q, where $P<Q$ and $P+Q=100$.

Group A consists of a fractions, where both P and Q are prime. Group B consists of b fractions which are in lowest terms, but at least one of P and Q is <u>not</u> prime.
Group C consists of c fractions which are not in lowest terms.

Compute the positive value of x that solves the proportion $\frac{a+c}{x}=\frac{x}{b}$.

MASSACHUSETTS MATHEMATICS LEAGUE
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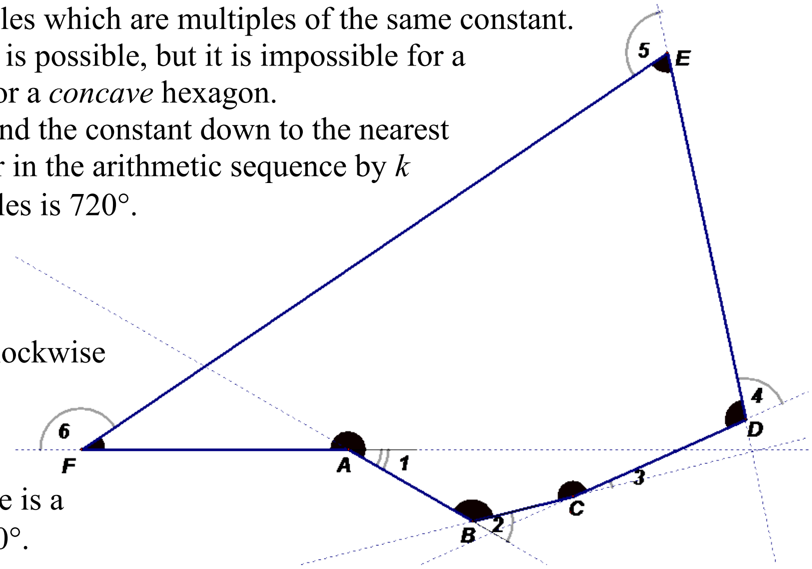
ROUND 7 TEAM QUESTIONS

F) An n-sided polygon has interior angles which are multiples of the same constant. For a triangle and quadrilateral, this is possible, but it is impossible for a pentagon. However, it is possible for a *concave* hexagon.
To eliminate tedious arithmetic, round the constant down to the nearest integer. Increase the largest number in the arithmetic sequence by k to insure the sum of the interior angles is 720°.
The diagram at the right shows such a hexagon. Let T be the sum of the measures of the *numbered* angles, where the angle measured clockwise (at A) is negative and other angles measured counterclockwise are positive. Note, since the hexagon is concave, at least one interior angle is a reflexive angle, i.e., greater than 180°.
<u>The diagram is drawn to scale.</u>
Compute the ordered pair (k,T).



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018 SOLUTION KEY

Team Round

A) The sides of triangles II and III are easy to get based only on special right triangles. ΔII is 5-12-13 and ΔIII is 13-84-85. Notice that in the following chart, the values in the first column increase by 2, the gaps between the values in the middle column are increasing by 4.

3	4	5
5	12	13
7	24	25
9	40	41
11	60	61
13	84	85
…		
85	?	?

Pursuing this pattern to find ΔIV would be painful, but the Pythagorean Theorem does quite nicely. $85^2 + x^2 = (x+1)^2 \Rightarrow 7225 = 2x+1 = x+(x+1)$

Thus, 7225 represents the long leg plus the hypotenuse.
All we need to do is add in 85 $\Rightarrow$**<u>7310</u>**.

B) $\dfrac{1}{b}\cdot\dfrac{c}{d} = \dfrac{10+c}{10b+d}$

Cross-multiplying, we have $10bc + cd = 10bd + bcd$

or $\dfrac{cd+10b(c-d)}{bcd} = 1 \Leftrightarrow \dfrac{1}{b} + \dfrac{10(c-d)}{cd} = 1$

Transposing terms, $1-\dfrac{1}{b} = \dfrac{b-1}{b} = \dfrac{10(c-d)}{cd}$ or $\boxed{\dfrac{b-1}{10b} = \dfrac{c-d}{cd}}$.

For $b = 2...9$, we have $\dfrac{b-1}{10b} = \mathbf{\dfrac{1}{20}}, \dfrac{2}{30} = \dfrac{1}{15}, \mathbf{\dfrac{3}{40}}, \dfrac{4}{50} = \dfrac{2}{25}, \dfrac{5}{60} = \mathbf{\dfrac{1}{12}} = \mathbf{\dfrac{2}{24}}, \dfrac{6}{70} = \dfrac{3}{35}, \dfrac{7}{80}, \dfrac{8}{90} = \mathbf{\dfrac{4}{45}}$.

Which can be written in the form $\dfrac{c-d}{cd}$?

$\mathbf{\dfrac{1}{20}} = \dfrac{5-4}{5\cdot4}, \mathbf{\dfrac{3}{40}} = \dfrac{8-5}{8\cdot5}, \mathbf{\dfrac{1}{12}} = \dfrac{4-3}{4\cdot3}, \mathbf{\dfrac{2}{24}} = \dfrac{6-4}{6\cdot4}, \mathbf{\dfrac{4}{45}} = \dfrac{9-5}{9\cdot5}$

The bolded fractions give rise to 5 ordered triples (b,c,d):

(2, 5, 4), (4, 8, 5), (6, 4, 3), (6, 6, 4), and (9, 9, 5)

Thus, $(k,m,M) = $ **<u>(5, 40, 405)</u>**.

<u>FYI</u>:

If a were not restricted to 1 (but $a < b$), the equality $\dfrac{a}{b}\cdot\dfrac{c}{d} = \dfrac{10a+c}{10b+d}$ would be valid for ordered quadruples (a, b, c, d) = (2, 6, 6, 5) and (4, 9, 9, 8), as well as (1, 2, 5, 4), (1, 4, 8, 5), (1, 6, 4, 3), (1, 6, 6, 4), and (1, 9, 9, 5).

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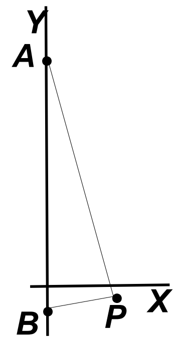
Team Round - continued

C) Solving the given system of equations $\begin{cases} 3x-2y=17 \\ 2x+3y=7 \end{cases}$ shows $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect at $P(5,-1)$.

With slopes of $\frac{3}{2}$ and $-\frac{2}{3}$, the given lines are perpendicular, so the angle bisectors are also perpendicular.The easiest way to find the equations of the angle bisectors is to simply add and subtract the given equations. The proof of this fact is included at the end of this solution key. You might want to investigate this assertion yourself (before peeking). The resulting equations are $\begin{cases} 5x+y=24 \\ x-5y=10 \end{cases}$ and the y-intercepts are at $A(0, 24)$ and $B(0, -2)$.



Thus, the triangle formed has an area of $\frac{1}{2}\cdot 5\cdot\left(24-{}^{-}2\right)=5\cdot 13=\underline{\mathbf{65}}$.

D) $f(x)=\log_8\left(4^{x^2}\cdot 16^x\cdot 64^k\right)\Leftrightarrow f(x)=x^2\log_8 4+x\log_8 16+k\log_8 64=\frac{2}{3}x^2+\frac{4}{3}x+2k$.

We require $\frac{2}{3}x^2+\frac{4}{3}x+2k=4\Leftrightarrow 2x^2+4x+6(k-2)=0\Leftrightarrow x^2+2x+3(k-2)=0$

$\Rightarrow x=\dfrac{-2\pm\sqrt{4-12(k-2)}}{2}=-1\pm\sqrt{1-3(k-2)}=-1\pm\sqrt{7-3k}$. $7-3k$ must be a perfect square.

$k=-1\Rightarrow 10$, $k=-2\Rightarrow 13$, $\boxed{k=-3\Rightarrow 16}\Rightarrow x=-1\pm 4$.Thus, $(k,x_1,x_2)=\underline{(\mathbf{-3,3,-5})}$.

E) There are 49 fractions satisfying $P<Q$ and $P+Q=100$, for positive integers.

Group A (P and Q both prime): $\left\{\frac{P}{Q}=\frac{3}{97},\frac{11}{89},\frac{17}{83},\frac{29}{71},\frac{41}{59},\frac{47}{53}\right\}$ (6 fractions)

Group C (not in lowest terms): Reducible fractions will either be

even/even: $\frac{2}{98},\frac{4}{96},\dots,\frac{48}{52}$ (24 fractions) or,

odd/odd: $\frac{5}{95},\frac{15}{85},\dots,\frac{45}{55}$ (5 fractions) - a total of 29 fractions.

Convince yourself that the common prime factor in this case can only be 5.

Group B must contain 49 – 6 – 29 = 14 fractions, all reduced, but either the numerator or the denominator (or both) are not prime.

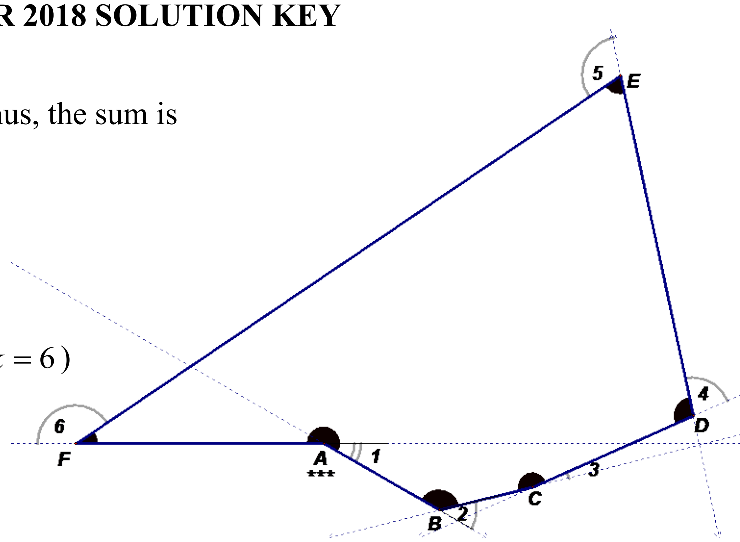
Enumerating the fractions was unnecessary, but, for confirmation, here they are:

$\frac{\mathbf{1}}{\mathbf{99}},\frac{7}{\mathbf{93}},\frac{\mathbf{9}}{\mathbf{91}},\frac{13}{\mathbf{87}},\frac{19}{\mathbf{81}},\frac{\mathbf{21}}{79},\frac{23}{\mathbf{77}},\frac{\mathbf{27}}{73},\frac{31}{\mathbf{69}},\frac{\mathbf{33}}{67},\frac{37}{\mathbf{63}},\frac{\mathbf{39}}{61},\frac{43}{\mathbf{57}},\frac{\mathbf{49}}{\mathbf{51}}$

The non-prime components are bolded.

$\frac{a+c}{x}=\frac{x}{b}\Leftrightarrow\frac{6+29}{x}=\frac{x}{14}\Rightarrow x^2=14(35)=2\cdot 5\cdot 7^2\Rightarrow x=\underline{\mathbf{7\sqrt{10}}}$.

Team Round - continued



F) The 6 interior angles are $c, 2c, 3c, ..., 6c$. Thus, the sum is

$$21c = 720 \Rightarrow c = \frac{240}{7} = 34^{+} \Rightarrow \lfloor c \rfloor = 34.$$

The sequence of angle measures is

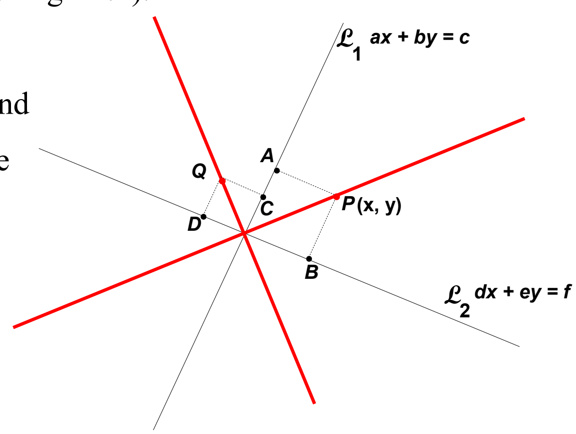
34, 68, 102, 136, 170, $\underbrace{\cancel{204}}_{210}$

The largest measure is changed to 210 ($\Rightarrow k = 6$) to insure the 6 interior angles sum to 720.The exterior angle at A(***) measures 150°, so $m\angle 1 = -30°$

$m\angle B = 136° \Rightarrow m\angle 2 = 44°$
$m\angle C = 170° \Rightarrow m\angle 3 = 10°$
$m\angle D = 102° \Rightarrow m\angle 4 = 78°$
$m\angle E = 68° \Rightarrow m\angle 5 = 112°$
$m\angle F = 34° \Rightarrow m\angle 6 = 146°$

Thus, $T = -30° + 44° + 10° + 78° + 112° + 146° = 360°$, and $(k, T) = \underline{(\mathbf{6, 360})}$.

FYI:
Surprising? To close the hexagon, we must turn "full circle", i.e. through 360°.
Let's travel around the figure counterclockwise (conventionally the "+" direction), starting at vertex *B* and returning to vertex *B*. At vertex *B*, the turn is counterclockwise from the baseline $\overline{AB}$ to reach the next vertex *C* along $\overline{BC}$.Similarly, for traveling from *C* to *D*, *D* to *E*, *E* to *F*, and *F* to *A*. However, traveling from *A* back to *B*, the turn is clockwise ("-"), because *A* was a reflexive angle! By adopting a convention which considers the direction of the turn, the sum of the turns becomes 360° (instead of a perplexing 420°).



Angle Bisectors Proof
Let*P*be a point on one of the angle bisectors of lines $\mathcal{L}_1$ and $\mathcal{L}_2$, $\overline{PA} \perp \mathcal{L}_1$, and $\overline{PB} \perp \mathcal{L}_2$. Then: $PA = PB$. Using the point-to-line distance formula,

$\frac{|ax + by - c|}{\sqrt{a^2 + b^2}} = \frac{|dx + ey - f|}{\sqrt{d^2 + e^2}}$. Since $\mathcal{L}_1 \perp \mathcal{L}_2$, the slopes are negative reciprocals, i.e., $\frac{a}{b} = -\frac{e}{d}$. For $\begin{cases} e = a \\ d = -b \end{cases}$, the denominators are equal and can be ignored.

$$|ax + by - c| = |dx + ey - f| \Leftrightarrow ax + by - c = \pm(dx + ey - f) \Leftrightarrow (a \pm d)x + (b \pm e) = (c \pm f).$$

Note that perpendicularity is a *sufficient condition*, but not a *necessary condition*.
If $a^2 + b^2 = d^2 + e^2$, adding and subtracting produces the equations of the angle bisectors!

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2018 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) $\frac{6}{5}$ B) $\left(-\frac{1}{288}, O\right)$ C) $3\sqrt{2}-\sqrt{6}$

Round 2 Arithmetic/Elementary Number Theory

A) 2, 5 B) (125, 75, 225) C) 9876352, 9875632

Round 3 Coordinate Geometry of Lines and Circles

A) (6, 10, 10) B) $\left(0, \frac{3}{2}\right), \frac{212\pi}{125}$ C) $(3,-6)$, $(-1,2)$

Round 4 Algebra 2: Log and Exponential Functions

A) $\sqrt{7}$ B) $\left(\frac{11}{8}, -\frac{7}{8}\right)$ C) $\frac{1}{9}$

Round 5 Algebra 1: Ratio, Proportion or Variation

A) $\frac{13}{15}$ B) 10^{-11} C) $13\left(3-\sqrt{5}\right)$

Round 6 Plane Geometry: Polygons (no areas)

A) 16 B) $(14+5\pi,\ 38+10\pi)$ C) 36

Team Round

A) 7310 D) $(-3, 3, -5)$

B) (5, 40, 405) E) $7\sqrt{10}$

C) 65 F) (6, 360)