MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2018
ROUND 1 COMPLEX NUMBERS (No Trig)

ANSWERS

A) (____________ , ____________)

B) ______________________________

C) ______________________________

A) $\sqrt{-2} \cdot \sqrt{-1} + \sqrt{-8} + \sqrt{-32} = a + bi$. Compute the ordered pair (a, b).

B) Given: $(1-i)^9 = x + yi$, where x and y are integer constants
Compute $\sqrt[4]{x-y}$.

C) D is an imaginary root of $x^3 + 8 = 0$.
Compute $(2 - 2D + D^2)(2 + 2D - D^2)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2018 SOLUTION KEY

Round 1

A) $\sqrt{-2}\cdot\sqrt{-1}+\sqrt{-8}+\sqrt{-32}=i\sqrt{2}\cdot i+i\sqrt{8}+i\sqrt{32}=i^2\sqrt{2}+2i\sqrt{2}+4i\sqrt{2}=-\sqrt{2}+\left(6\sqrt{2}\right)i$

Thus, $(a,b)=\underline{\left(-\sqrt{2},6\sqrt{2}\right)}$.

B) $(1-i)^9=\left((1-i)^2\right)^4(1-i)=(1-2i-1)^4(1-i)=(-2i)^4(1-i)=16-16i$.

Thus, $\sqrt[4]{x-y}=\sqrt[4]{32}=\underline{\mathbf{2\sqrt[4]{2}}}$.

C) $x^3+8=0\Leftrightarrow(x+2)\left(x^2-2x+4\right)=0$. Since $x=D$ is a root (from the trinomial factor), we have $D^2-2D+4=0\Leftrightarrow\boxed{D^2+4=2D}$. Thus,

$\left(2-2D+D^2\right)\left(2+2D-D^2\right)=\left(2-\left(D^2+4\right)+D^2\right)\left(2+\left(D^2+4\right)-D^2\right)=(-2)(6)=\underline{\mathbf{-12}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2018
ROUND 2 ALGEBRA 1: ANYTHING

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Let the constant A be the largest odd divisor of 576.
Let the constant B be the smallest positive 3-digit multiple of 7
Let the constant C be the second smallest positive 2-digit number which is 1 more than a power of 2.
Solve the equation $Ax + B = C$.

B) At a sports bar, the ratio of people to stools is 2 : 5.
The total number of "legs" is 76.
Assume people are two-legged and stools are thee-legged.
If all the people sit at the bar, how many unoccupied stools are there?

C) $P(-7,4), Q\left(-\frac{3}{2}, y\right)$, and $R(x,10)$ are collinear and $PQ = QR$.
Compute k, if point $S(k,k)$ lies on the line passing through P, Q and R.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2018 SOLUTION KEY

Round 2

A) $576 = 4 \cdot 144 = 16 \cdot 36 = 2^6 3^2$
Since the largest odd factor takes all of the odd prime factors and none of the even, $A = 9$.
Since $7 \cdot 14 = 98$, the smallest 3-digit multiple of 7 is $B = 105$.
Since $2^3 = 8$, $2^4 = 16$, and $2^5 = 32$, $C = 33$.
$9x + 105 = 33 \Rightarrow x = \dfrac{33-105}{9} = \dfrac{-72}{9} = \underline{\mathbf{-8}}$.

B) Suppose there are k people and N stools.
$k : N = 2:5 \Rightarrow k = 2a,\ N = 5a$, for some integer a.
$2k + 3N = 76 \Rightarrow 4a + 15a = 76 \Rightarrow a = 4 \Rightarrow (k, N) = (8, 20) \Rightarrow N - k = \underline{\mathbf{12}}$.

C) Given collinear points $P(-7,4), Q\left(-\dfrac{3}{2}, y\right)$, and $R(x,10)$.
Since Q is the midpoint of $\overline{PR}$, $-\dfrac{3}{2} = \dfrac{x+(-7)}{2}, y = \dfrac{4+10}{2} \Rightarrow (x, y) = (4,7)$.
Therefore, the slope of the line $\overleftrightarrow{PR}$ is $m = \dfrac{10-4}{4-(-7)} = \dfrac{6}{11}$ and, using point P, the equation in point-slope form is $y - 4 = \dfrac{6}{11}(x+7)$. Letting $x = y = k$ and clearing fractions, we have
$11(k-4) = 6(k+7) \Rightarrow 5k = 42 + 44 = 86 \Rightarrow k = \underline{\mathbf{17.2}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2018
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

ANSWERS

A) ________________________________

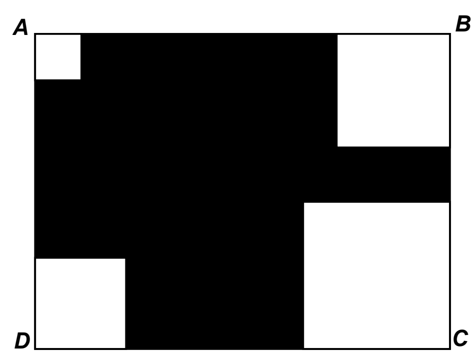
B) ________________________________

C) (________ , ________ , ________)

A) Consider a square with side of length x and a rectangle with sides of length x and $x - k$
For how many ordered pairs (k, x), where k and x are integers, will the area of the square exceed the area of the rectangle by 48 square units?

B) $\overline{PQ}$ is the median in trapezoid $ABCD$, where $\overline{AB} \parallel \overline{CD}$, $CD > AB$
E is a point on $\overline{CD}$ such that $\overline{PE} \perp \overline{CD}$.
If $PE = 6$ and $PQ = 7.5$, compute the area of $ABCD$.

C) Rectangle $ABCD$ has area 210 square units and dimension m by n units, where m and n are integers and $m < n$.
As indicated, squares are cut out of each corner.
The sides of the squares at A, B, C, and D are x, $x + 2$, $x + 3$ and $x + 1$ respectively.
The area of the shaded region is half the area of $ABCD$.
Compute the ordered triple (x, m, n).

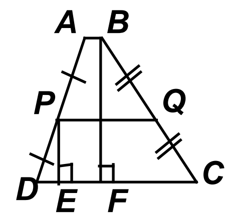


MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2018 SOLUTION KEY

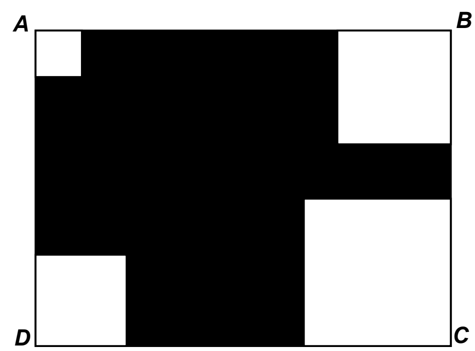
Round 3

A) $x^2 - x(x-k) = 48 \Rightarrow kx = 48 \Rightarrow (k,x) = (1,48),(2,24),(3,16),(4,12),(6,8)$
Thus, there are **<u>5</u>** possible ordered pairs.

B) Since the length of the median in any trapezoid is the average of the lengths of the bases,
area $= \frac{1}{2}h(b_1 + b_2) = h \cdot \frac{b_1 + b_2}{2} = hm$.
$PE = 6 \Rightarrow BF = h = 12$
Therefore, $area = 12(7.5) = \underline{\mathbf{90}}$.



C) The sum of the areas of the 4 squares is
$x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2 = 105 \Leftrightarrow 4x^2 + 12x + 14 = 105$
$\Rightarrow 4x^2 + 12x - 91 = 0 \Leftrightarrow (2x-7)(2x+13) = 0 \Rightarrow x = 3.5$
The possible dimensions of *ABCD* are 1 x 210, 2 x 105,
3 x 70, 5 x 42, 6 x 35, 7 x 30, and 14 x 15.
The only one which allows the 4 cutouts is 14 x 15.
Therefore, $(x,m,n) = \underline{(\mathbf{3.5,14,15})}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2018
ROUND 4 ALG 1: FACTORING AND ITS APPLICATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Compute the *minimum* positive integer k for which $6k$ has more than 10 factors.

B) For some positive integers A, $x^4 + Ax^2 + 625$ can be written as the difference of perfect squares, namely $\left(x^2+25\right)^2 - \left(Nx\right)^2$, where N is also a positive integer.
Determine the sum of all possible values of A.

C) Solve for x. $$x^6 + \left(42x\right)^2 \le 85x^4$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2018 SOLUTION KEY

Round 4

A) We can count factors without listing them, e.g., $24 = 2^3 3^1$ has $(3+1)(1+1) = 8$ factors, since factors of 24 can only have prime factors of 2 and 3, and the exponent of 2 can only be 0,1,2, or 3, and the exponent of 3 can only be 0 or 1. In general, the exponents can range from 0 to the maximum number of times the prime factor occurred in the prime factorization of *N*.

$6k$	Factorization	# factors
6	$2^1 3^1$	4
12	$2^2 3^1$	6
18	$2^1 3^2$	6
24	$2^3 3^1$	8
30	$2^1 3^1 5^1$	8
36	$2^2 3^2$	9
42	$2^1 3^1 7^1$	8
48	$2^4 3^1$	10
54	$2^1 3^3$	8
60	$2^2 3^1 5^1$	12

$\Rightarrow k = \underline{\mathbf{10}}$.

B) $\left(x^2+25\right)^2 - (Nx)^2 = x^4 + \left(50 - N^2\right)x^2 + 625$

Thus, $A = 50 - N^2$. For integers N = 1, 2, …, 7, A is also a positive integer, namely 49, 46, 41, 34, 25, 14 and 1.
Grouping advantageously, we have (49 + 41) + (46 + 34) + (25 + 14 + 1) = **<u>210</u>**.

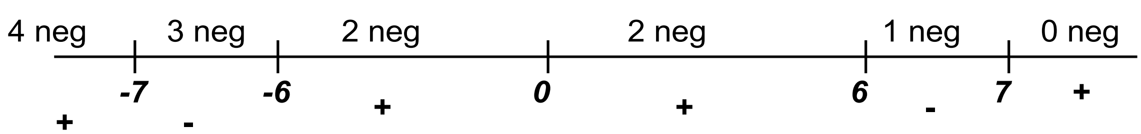
C) $x^6 + (42x)^2 < 85x^4 \Leftrightarrow x^2\left(x^4 - 85x^2 + 42^2\right) < 0$

Examining the trinomial piece, we have

$$x^4 - 85x^2 + 42^2 = \left(x^4 - 84x^2 + 42^2\right) - x^2 = \left(x^2 - 42\right)^2 - (x)^2 = \left(x^2 + x - 42\right)\left(x^2 - x - 42\right)$$

$$= (x+7)(x-6)(x-7)(x+6)$$

Thus, the critical points are at $0, \pm 6, \pm 7$ and the sign of the product is determined by the number of negative factors.



There is no sign change at $x = 0$, since the factor x occurs twice $\left(x^2 = (x-0)^2\right)$.

Thus, the solution is $\underline{\boldsymbol{-7 \le x \le -6} \text{ or } \boldsymbol{x = 0} \text{ or } \boldsymbol{6 \le x \le 7}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2018
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES

ANSWERS

A) ______________________________

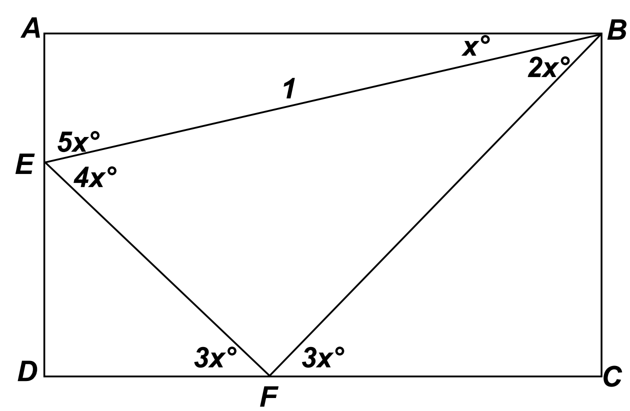
B) ______________________________

C) ______________________________

A) Compute $\cos\left(\dfrac{n\pi}{6}\right)$, if n is the *smallest* prime greater than 89.

B) The lines $y = \sin 30°$ and $x = \tan 45°$ intersect at point $A(x_1, y_1)$.
$y = \cos x$ reaches a maximum at $B(x_2, y_2)$, where $x_2 \cdot y_2$ is a rational number.
Compute the distance between A and B.

C) $ABCD$ is a rectangle. $BE = 1$ and angle measures are as indicated. Compute AE. Your answer may not contain any radical whose radicand is an irrational number.



MASSACHUSETTS MATHEMATICS LEAGUE
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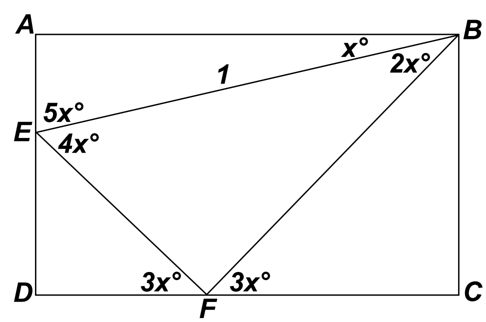
Round 5

A) Since $91 = 7 \cdot 13$, 93 $(\div 3)$ and 95 $(\div 5)$ are composite, we test 97 (by prime divisors $< \sqrt{97}$).
2, 3, 5, and 7 each leave a nonzero remainder, guaranteeing that 97 is prime.
Since $\frac{97\pi}{6} = \pi\left(16\frac{1}{6}\right) = 16\pi + \frac{\pi}{6} = 8(2\pi) + \frac{\pi}{6}$, $\frac{97\pi}{6}$ is coterminal with $\frac{\pi}{6}$.
Trig functions of coterminal values always have equal values.
$\therefore \cos\left(\frac{97\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \underline{\boldsymbol{+\frac{\sqrt{3}}{2}}}$.

B) $\begin{cases} y = \sin 30° \Leftrightarrow y = \frac{1}{2} \\ x = \tan 45° \Leftrightarrow x = 1 \end{cases} \Rightarrow A\left(1, \frac{1}{2}\right)$

$y = \cos x$ attains is maximum of 1 at $x = 0$ and at intervals of 2π , both right and left.
$\Rightarrow (0,1), (\pm 2\pi, 1), (\pm 4\pi, 1), \ldots$
The only point at which $x_2 \cdot y_2$ is rational is at $B(0,1)$.
Thus, $AB = \sqrt{(1-0)^2 + \left(\frac{1}{2} - 1\right)^2} = \sqrt{\frac{5}{4}} = \underline{\boldsymbol{\frac{\sqrt{5}}{2}}}$.



C) In ΔABE , $6x = 90 \Rightarrow x = 15$ and ΔBFE is a $30° - 60° - 90°$ (right) triangle with a hypotenuse of length 1.
Therefore, $EF = \frac{1}{2}$ and $BF = \frac{\sqrt{3}}{2}$.
Since both ΔDEF and ΔBCF are isosceles right triangles,
we have $DF = DE = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{4}$ and
$FC = BC = AD = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{6}}{4}$. The required length is $AE = AD - ED = \underline{\boldsymbol{\frac{\sqrt{6} - \sqrt{2}}{4}}}$.
Note: Since the hypotenuse of ΔABE is 1, this means that the exact value of
$\sin 15° = \frac{\sqrt{6} - \sqrt{2}}{4}$.

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS

ANSWERS

A) (____________ , ____________)

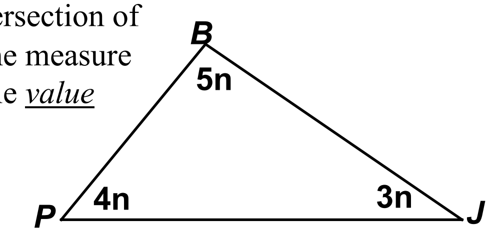
B) ____________________________

C) ____________________________

A) The measures of the angles in a scalene triangle are in a 7:10:13 ratio.
The altitude from the vertex of the largest angle divides that angle into an $a : b$ ratio, where $a < b$. Compute the ordered pair of integers (a,b).

B) $\Delta ADE \sim \Delta ABC$, where D lies on $\overline{AC}$, E lies on $\overline{AB}$, and $\overline{DE}$ is *not* parallel to $\overline{BC}$.
The measures of angles A, B, and C (in some order) form an arithmetic sequence with a common difference of 12.
Draw a line through point E parallel to $\overline{BC}$, intersecting $\overline{AC}$ in point F.
Compute <u>all</u> possible degree measures of $\angle FED$.

C) Let α denote the measure of the obtuse angle formed by intersection of the angle bisectors of two angles in ΔPBJ. Let β denote the measure of the third angle in ΔPBJ. Determine the <u>smallest</u> possible <u>*value*</u> of the ratio $\dfrac{\alpha}{\beta}$, rounded to the nearest tenth, if necessary.



MASSACHUSETTS MATHEMATICS LEAGUE
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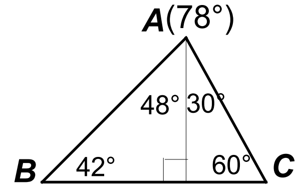
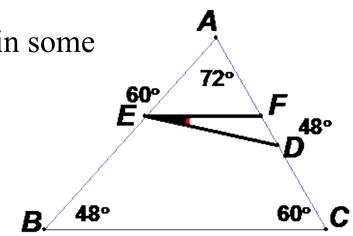
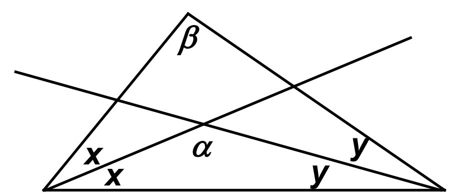
Round 6



A) $7x+10x+13x=180 \Rightarrow x=6$.
Thus, the largest angle is 78°.
The altitude creates two right triangles containing acute angles of 42° and 60°. The other acute angles into which the largest angle of the given scalene are divided are 48° and 30°. Therefore, the ratio is 30 : 48 = 5 : 8 and $(a,b)=\underline{\mathbf{(5,\ 8)}}$.



B) $(a-12)+a+(a+12)=180 \Rightarrow a=60 \Rightarrow \{A,B,C\}=\{72°,60°,48°\}$ in some order. Depending on the locations of the three angle measures, $m\angle FED = 72-60,\ 72-48,$ or $60-48$.
Thus, the only possible angle measures are <u>**12**</u> and <u>**24**</u>.
The degree symbol is not necessary.



C) $12n=180 \Rightarrow n=15 \Rightarrow P(45°),\ B(60°),\ J(\ 75°)$.
β, the non-bisected angle could be P, B or J.

$$P\text{: } \beta=45°,(x,y)=(30,37.5) \Rightarrow \alpha=112.5 \Rightarrow \frac{\alpha}{\beta}=\frac{225}{90}=\frac{25}{10}=2.5.$$

$$B\text{: } \beta=60°,(x,y)=(37.5,22.5) \Rightarrow \alpha=120 \Rightarrow \frac{\alpha}{\beta}=2.$$

$$J\text{: } \beta=75°,(x,y)=(30,22.5) \Rightarrow \alpha=127.5 \Rightarrow \frac{\alpha}{\beta}=\frac{255}{150}=\frac{51}{30}=\underline{\mathbf{1.7}}.$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2018
ROUND 7 TEAM QUESTIONS

ANSWERS

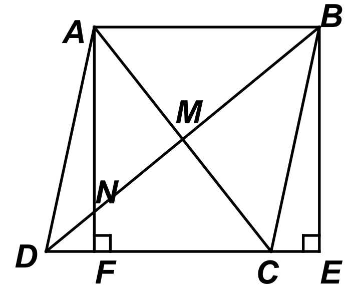
A) ______________________________ D) (______ , ______ , ______ , ______)

B) ______________________________ E) ______________________________

C) ______________________________ F) ______________________________

A) Let P_1 be the point (a, b) in the complex plane corresponding to $a+bi=4+0i$.
Let P_n be the point in the complex plane corresponding to $P_{n-1}\cdot\dfrac{1+i}{\sqrt{2}}$, for $n>1$.
This sequence of points determines a polygon whose area is K units^2. Compute K.

B) Consider the sequence of integral Pythagorean triplets (a,b,c), where a increases by 2, and $c=b+1$. $P_1=(3,4,5)$, $P_2=(5,12,13)$, $P_3=(7,24,25)$,...
Compute the area of the right triangle specified by P_{49}.



C) In rhombus $ABCD$, $BD=24$, $BC=16$, $DC:CE=8:1$ and $\overline{AF}$, $\overline{BE}\perp\overline{DC}$
Compute the area of ΔFND.

D) Let $N=x^2+xy-6y^2-8x+6y+12$
If $x=y$, then $N=0$ for $x=a$ and $x=b$, where $a<b$.
If $N=0$ and $y=t=a+b$, then $x=c$ or $x=d$, where $c<d$.
Compute the ordered quadruple (a,b,c,d).

E) Using the letters A to E, arrange the following from smallest to largest:
$A=\sin(-750°)$ $B=\cos(-150°)$ $C=\tan(840°)$ $D=\sec(421°)$ $E=\csc(-335°)$

F) An isosceles triangle is called "exceptional", if and only if two of its angles have measures x and kx, where x and k are natural numbers and $k>1$. An equivalence class of triangles, denoted $E(T)$, contains all triangles which are similar to ΔT. Compute the number of distinct equivalence classes of exceptional triangles.

MASSACHUSETTS MATHEMATICS LEAGUE
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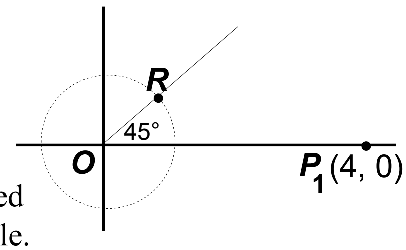
Team Round



A) $\frac{1+i}{\sqrt{2}}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ corresponds to a point R in the complex plane on the unit circle centered at the origin (alternately referred to as the pole). Points R, the origin and P_1 determine a 45° angle.

Multiplying by $\frac{1+i}{\sqrt{2}}$ "rotates" a given point 45° around the origin (in this case around a circle of radius 4). Thus, the P_n sequence determines a regular octagon with radius 4. Subdividing the octagon into 8 isosceles triangles, we have an area of

$8\left(\frac{1}{2}\cdot 4\cdot 4\cdot \sin 45°\right)=64\cdot\frac{\sqrt{2}}{2}=\underline{\mathbf{32\sqrt{2}}}$.

Converting from $(a + bi)$ to $(rcis\theta)$ and applying DeMoivre's Theorem provides confirmation.

$4+0i=4cis0°,\ \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i=cis45°$, and $P_n=4cis0°\cdot\left(cis45°\right)^{n-1}=4cis\left(45°\left(n-1\right)\right)$

$\Rightarrow\ P_2=4cis45°,\ P_3=4cis90°,\ldots,\ P_8=4cis315°,\ P_9=4cis360°=P_1$

Thus, we have 8 equally spaced points around a circle of radius 4, vertices of a regular octagon.

B)

n	a	b	c	S	1st Diff	2nd Diff	3rd Diff
1	3	4	5	6			
2	5	12	13	30	24		
3	7	24	25	84	54	30	
4	9	40	41	180	96	42	12
5	11	60	61	330	150	54	12
6	13	84	85	546	216	66	12

Since the third differences are the same, the general formula is 3rd degree, i.e., of the form $S=An^3+Bn^2+Cn+D$. Substituting,

$$\left.\begin{array}{l} n=1:\ A+B+C+D=6 \\ n=2:\ 8A+4B+2C+D=30 \\ n=3:\ 27A+9B+3C+D=84 \\ n=4:\ 64A+16B+4C+D=180 \end{array}\right\}\Rightarrow\begin{cases} 7A+3B+C=24 \\ 19A+5B+C=54 \\ 37A+7B+C=96 \end{cases}$$

$$\Rightarrow\begin{cases} 12A+2B=30 \\ 18A+2B=42 \end{cases}\Rightarrow 6A=12\Rightarrow\left(A,B\right)=\left(2,3\right)$$

$C=1,\ D=0\Rightarrow P\left(n\right)=2n^3+3n^2+n=n\left(n+1\right)\left(2n+1\right)$.

$\therefore\ P\left(49\right)=49\cdot 50\cdot 99=2450\left(100-1\right)\Rightarrow\begin{cases} \ \ 245000 \\ \underline{-\ \ \ \ 2450} \end{cases}\Rightarrow\underline{\mathbf{242{,}550}}$.

Team Round - continued

C) Let $DF = CE = x$ and $BE = AF = y$.

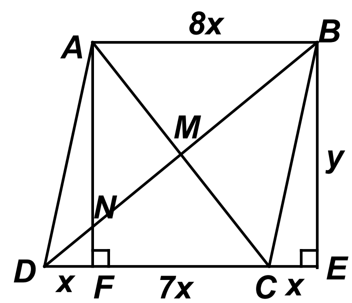
In ΔBED, $(9x)^2 + y^2 = 24^2 = 576$.

In ΔBEC, $x^2 + y^2 = 16^2 = 256$.

Subtracting, $80x^2 = 320 \Rightarrow x = 2,\ y = 6\sqrt{7}$.

Since $\Delta FND \sim \Delta ANB$ and $\dfrac{DF}{BA} = \dfrac{FN}{AN} = \dfrac{1}{8}$,

$|\Delta FND| = \dfrac{1}{2} \cdot FD \cdot FN = \dfrac{1}{2} x\left(\dfrac{1}{9} AF\right) = \dfrac{xy}{18} = \dfrac{2\left(6\sqrt{7}\right)}{18} = \underline{\boldsymbol{\dfrac{2}{3}\sqrt{7}}}$.



D) Let $x = y = k$. Then:

$$k^2 + k^2 - 6k^2 - 8k + 6k + 12 = -4k^2 - 2k + 12 = -2\left(2k^2 + k - 6\right) = -2(2k-3)(k+2) = 0 \Rightarrow k = \frac{3}{2}, -2$$

Therefore, $(a,b) = \left(-2, \dfrac{3}{2}\right) \Rightarrow\ y = t = -\dfrac{1}{2}$.

Substituting for y is only slightly painful, and we could proceed as above and find the corresponding x-values, but for <u>nastier</u> y-values, finding the corresponding x-values could become very tedious. Another approach might be to set N equal to 0, factor N, treating the expression as a quadratic (in x), namely, $N = x^2 + (y-8)x - 6\left(y^2 - y - 2\right) = 0$.

Using the quadratic formula, we have $x = \dfrac{(8-y) \pm \sqrt{(y-8)^2 + 24\left(y^2 - y - 2\right)}}{2}$

This may look awful, but let's examine the discriminant.

$(y-8)^2 + 24\left(y^2 - y - 2\right) = 25y^2 - 40y + 16 = (5y-4)^2$ and the radical disappears.

$$x = \frac{(8-y) \pm (5y-4)}{2} = \frac{4y+4}{2}, \frac{-6y+12}{2} = 2y+2,\ 6-3y$$

$$\begin{cases} \boxed{x = 2y+2} \text{ or } y = \dfrac{x-2}{2} \text{ or } x - 2y - 2 = 0 \\ \boxed{x = 6-3y} \text{ or } y = \dfrac{6-x}{3} \text{ or } x + 3y - 6 = 0 \end{cases}$$

(An aside: Notice that we have factored the given expression as $(x-2y-2)(x+3y-6)$!!)

The boxed expressions make it easy to find x for a given y-value.

$y = -\dfrac{1}{2} \Rightarrow x = 1, \dfrac{15}{2} \Rightarrow (c,d) = \left(1, \dfrac{15}{2}\right)$ and the required ordered quadruple is $\underline{\boldsymbol{\left(-2, \dfrac{3}{2}, 1, \dfrac{15}{2}\right)}}$.

Team Round - continued

E) Simplifying and evaluating, where possible, we have

$$A = \sin\left(-750°\right) = \sin(-30°) = -\frac{1}{2}$$

$$B = \cos\left(-150°\right) = \cos(150°) = -\frac{\sqrt{3}}{2}$$

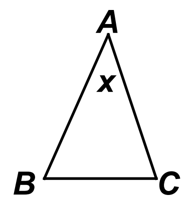
$$C = \tan\left(840°\right) = \tan(120°) = -\sqrt{3}$$

$$D = \sec(421°) = \sec(61°) = \csc(29°) = \frac{1}{\sin(29°)}$$

$$E = \csc(-335°) = \csc(25°) = \frac{1}{\sin(25°)}.$$

Clearly, D and E are positive values and, since the sine is an increasing function in the first quadrant, sin(29°) > sin(25°) > 0 and taking the reciprocals, $E > D$.
Thus, the correct order is **<u>CBADE</u>**.

F) Case 1: Suppose the vertex angle is x° and each base angle is (kx)°



Then: $x + 2\left(kx\right) = x\left(2k+1\right) = 180 \Rightarrow x = \frac{180}{2k+1}$

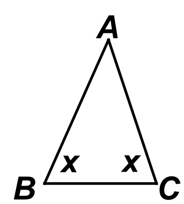
Clearly, we require $2k + 1$ be an ODD factor of 180.

Since $180 = 2^2\left(3^2 5^1\right)$, we see that 180 has 18 factors and 6 of them are odd, namely, 1, 3, 5, 9, 15, 45. [1 (k = 0) and 3 (k = 1) are eliminated, since we are given $k > 1$.)

$2k+1 = 5, 9, 15, 45 \Rightarrow k = 2, 4, 7, 22 \Rightarrow\ x = \underline{36, 20, 12, 4}$ (4 values)

$\Rightarrow\ \left(A, B, C\right) = \left(36, 54, 54\right), \left(20, 70, 70\right), \left(12, 84, 84\right), \left(4, 88, 88\right)$

Case 2: Suppose the vertex angle is (kx)° and each base angle is x°



Then: $kx + 2\left(x\right) = x\left(k+2\right) = 180 \Rightarrow x = \frac{180}{k+2}$

Clearly, we require that $k + 2$ be a factor of 180, and it can be either even or odd.

Since $k + 2 \neq 1, 2, 3$, we set up the following table to track the remaining 15 factors of 180.

We are interested only in whether there are duplicates, and completing the table is not necessary to verify that there are none and that the answer is 4 + 15 = **<u>19</u>**.

The table is included as an **FYI**.

$k+2$	4	5	6	9	10	12	15	18	20	30	36	45	60	90	180
k	2	3	4	7	8	10	13	16	18	28	34	43	58	88	178
x	45	36	30	20	18	15	12	10	9	6	5	4	3	2	1
kx	90	108	120	140	144	150	156	160	162	168	170	172	174	176	178

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2018 ANSWERS

Round 1 Algebra 2: Complex Numbers (No Trig)

A) $\left(-\sqrt{2}, 6\sqrt{2}\right)$ B) $2\sqrt[4]{2}$ C) -12

Round 2 Algebra 1: Anything

A) -8 B) 12 C) 17.2

Round 3 Plane Geometry: Area of Rectilinear Figures

A) 5 B) 90 C) $(3.5, 14, 15)$

Round 4 Algebra: Factoring and its Applications

A) 10 B) 210 C) $-7 \le x \le -6,\ \ x = 0,\ \ 6 \le x \le 7$

Round 5 Trig: Functions of Special Angles

A) $\dfrac{\sqrt{3}}{2}$ B) $\dfrac{\sqrt{5}}{2}$ C) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

Round 6 Plane Geometry: Angles, Triangles and Parallels

A) (5, 8) B) 12°, 24° C) 1.7

Team Round

A) $32\sqrt{2}$ D) $\left(-2, \dfrac{3}{2}, 1, \dfrac{15}{2}\right)$

B) 252,540 E) CBADE

C) $\dfrac{2}{3}\sqrt{7}$ F) 19