MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018
ROUND 1 VOLUME & SURFACES

ANSWERS

A) ____________________________ m^3

B) __________________________ $units^3$

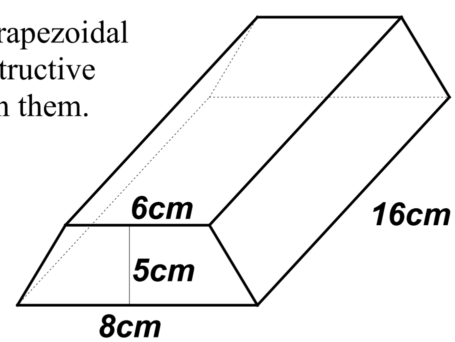
C) ____________________________ lbs.

A) A water tank, in the shape of an inverted right circular cone, has a diameter of 12 meters and a height of 10 meters. If the height of the water in the tank is 3 meters, compute the exact amount of water (in m^3) in the tank.

B) A pyramid has its vertex at the center of a face of a cube, and its base coincides with the opposite face of the cube. If the volume of the region <u>inside</u> the cube and <u>outside</u> the pyramid is 1152 $units^3$, compute the edge of the cube.

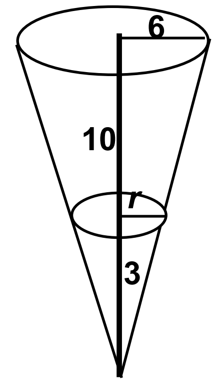
C) Fake gold bars, like those in the diagram at the right (an isosceles trapezoidal prism), are made of lead, spray-painted gold. The simplest non-destructive method for distinguishing a fake bar from a genuine one is to weigh them. Gold has a density of 19.3 g/cc (grams per cubic centimeter). Lead has a density of 11.3 g/cc.

In a shipment of 6 bars, 5 of which are fake, how much less than expected would the shipment weigh to the nearest tenth of a pound (lb)? One kilogram (1000 grams) is equivalent to 2.2 pounds.



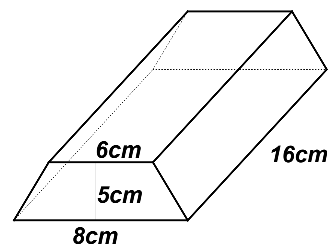
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018 SOLUTION KEY

Round 1



A) By similar triangles, $\frac{6}{10} = \frac{r}{3} \Rightarrow r = \frac{9}{5} \Rightarrow V = \frac{1}{3}\pi\left(\frac{9}{5}\right)^2(3) = \underline{\mathbf{\frac{81\pi}{25}}}$ (or $\underline{\mathbf{3.24\pi}}$).

B) Let x denote the edge of the cube. Then the volume of the required region is
$x^3 - \frac{1}{3}x^2 \cdot x = \frac{2}{3}x^3 = 1152 = 8(144) = 2^3\left(2^4 \cdot 3^2\right) \Rightarrow x^3 = 2^6 \cdot 3^3 \Rightarrow x = 4 \cdot 3 = \underline{\mathbf{12}}$.



C) The volume of the gold bar is $\frac{1}{2}5(6+8)16 = 70 \cdot 8 = 560$ cm^3.
Thus, each fake bar decreases the weight of the shipment by $560(19.3-11.3) = 4480$ grams.
Our shipment is short $5(4480) = 22,400$ grams = 22.4 kg.
To the nearest 0.1 lb., this is equivalent to $22.4(2.2) = 49.28 \Rightarrow \underline{\mathbf{49.3}}$ lbs.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

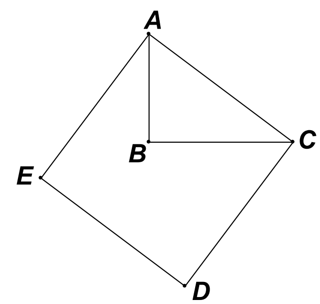
ANSWERS

A) ______________________________

B) ______________________________

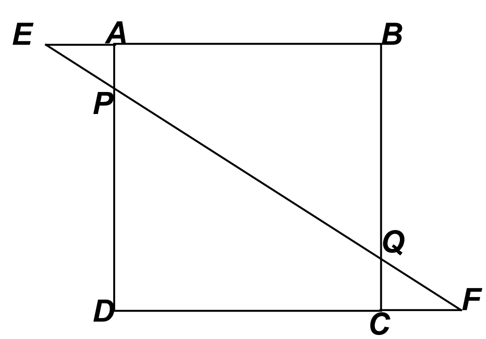
C) ______________________________

A) $AC = BC + 1 = AB + 2$, $\angle BAC$ and $\angle BCA$ are complementary.
$AE = DE = DC$, $\overline{AE} \perp \overline{DE}$, $\overline{DE} \perp \overline{DC}$
Compute the area of the concave pentagon *ABCDE*.



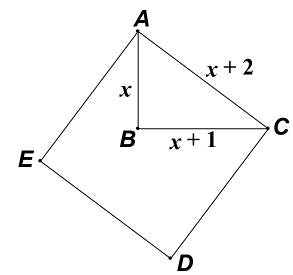
B) In ΔABC, $m\angle A = 60°$, $a = 16$, $b = 18$, and $\angle B$ is obtuse.
Compute *AB*.
Note: Lowercase letters denote the length of sides opposite angles whose vertex is denoted by the corresponding uppercase letter.

C) Given: $ABCD$ is a square, $AE = CF$
$AP = 1$ and $BQ = 7$
$\{A, B, E\}$ and $\{C, D, F\}$ are sets of collinear points.
Compute *PQ*.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018 SOLUTION KEY

Round 2



A) Is ΔABC a 3-4-5 right triangle?
We were *not* given that AC was an integer!

$(x+2)^2 = x^2 + (x+1)^2$

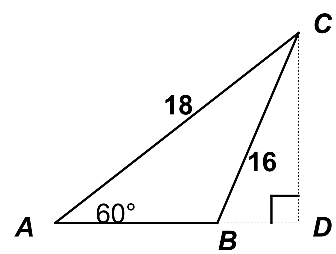
$4x + 4 = x^2 + 2x + 1$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, \cancel{-1}$

ΔABC must be 3-4-5.
$ACDE$ is a square with side 5.
Thus, the area of $ABCDE$ is $5^2 - \frac{1}{2} \cdot 3 \cdot 4 = \underline{\mathbf{19}}$.

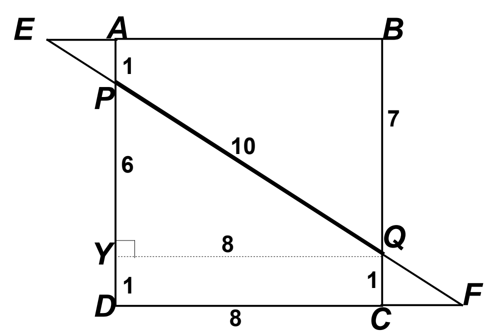


B) $DC = 9\sqrt{3},\ AD = 9$

$BD^2 + \left(9\sqrt{3}\right)^2 = 16^2$

$BD^2 = 256 - 243 = 13$

$AB = \underline{\mathbf{9 - \sqrt{13}}}$.



C) $\Delta PEA \cong \Delta QFC$ (by A.S.A.) $\Rightarrow (QC, QB) = (1, 7) \Rightarrow$
$BC = AB = 8$
Let $AE = CF = x$. Then:

$\Delta PEA \sim \Delta QEB \Rightarrow \frac{x}{x+8} = \frac{1}{7} \Rightarrow x = \frac{4}{3}$.

Using the Pythagorean Theorem on ΔPEA,

$PE^2 = 1^2 + \left(\frac{4}{3}\right)^2 = \frac{9+16}{9} \Rightarrow PE = \frac{5}{3}$.

Finally, $\frac{\frac{5}{3}}{\frac{5}{3} + PQ} = \frac{1}{7} \Rightarrow \frac{35}{3} = \frac{5}{3} + PQ \Rightarrow PQ = \frac{30}{3} = \underline{\mathbf{10}}$.

Alternately, by drawing $\overline{YQ} \parallel \overline{DC}$, ΔPYQ is a 3-4-5 right triangle
Therefore, $YQ = \underline{\mathbf{10}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Quinn has 4 stamp collections. In three of them he has a total of 1600 stamps. The fourth collection contains 55 fewer stamps than the average number of stamps in all four collections. How many stamps are in Quinn's fourth collection?

B) If $\frac{a}{b}=\frac{3}{7}, \frac{c}{b}=\frac{5}{6}$, and $a+c=1166$, compute b.

C) Two equal-volume well-insulated containers of water are to be mixed.
Container #1 holds water at a constant temperature of 60°.
Container #2 holds water at a constant temperature of 208°.
$(x+8)\%$ of the first container and $x\%$ of the second container are mixed.
The temperature of the mixture is to be 98.6°.
Assuming no heat loss during the transfer and mixing, compute $x\%$ of x.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018 SOLUTION KEY

Round 3

A) Let x denote the number of stamps in the fourth collection. Then:

$$x = \frac{1600 + x}{4} - 55 \Leftrightarrow 4x = (1600 + x) - 220 \Leftrightarrow 3x = 1380 \Leftrightarrow x = \underline{\mathbf{460}}.$$

B) Since $\frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c} = \frac{3}{7} \cdot \frac{6}{5} = \frac{18}{35}$, let $(a, c) = (18n, 35n)$.

$$a + c = 53n = 1166 \Rightarrow n = 22 \Rightarrow a = 18 \cdot 22 = (20+2)(20-2) = 400 - 4 = 396$$

$$\frac{396}{b} = \frac{3(132)}{b} = \frac{3}{7} \Rightarrow b = 7(132) = \underline{\mathbf{924}}.$$

C) Note $x\% = \frac{x}{100}$. It is required that $\frac{x+8}{100} \cdot 60 + \frac{x}{100} \cdot 208 = 98.6$.

$$\Rightarrow (60 + 208)x + 480 = 9860 \Rightarrow x = \frac{9380}{268} = \frac{4 \cdot 67 \cdot 35}{4 \cdot 67} = 35$$

35% of 35 $\Rightarrow .35(35) = \underline{\mathbf{12.25}}$.

FYI: Squaring a number ending in 5 is easy. $(\underline{X}\,\underline{5})^2$ always ends in 25.

The remaining digits will be $X(X+1)$.

Thus, $3 \cdot 4 = 12$ affixed to 25 $\Rightarrow$ 1225 and we adjust the decimal point.

This works even if $X > 9$! For example, a calculator easily verifies that $195^2 = 38025$.
But now you don't need a calculator.
Can you argue why this shortcut works?
If you try and come up short, ask your coach or send me an email @ olson.re@gmail.com.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

ANSWERS

A) ______________________________

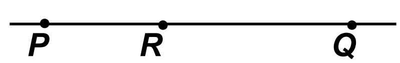
B) ______________________________

C) ______________________________

A) If the fractions $\frac{4}{25}, \frac{3}{20}, -\frac{3}{4}, \frac{7}{40}, \frac{7}{10}$ are arranged in increasing order, compute the decimal equivalent of the fraction in the middle of the list.

B) The length and width of a 4 x 6 rectangle are each increased by x units.
The perimeter of the resulting rectangle is 40% larger than that of the original rectangle.
By what percent has the area increased?

C) Points P, Q, and R are points on the x-axis.
The x-coordinate of P is $\frac{22}{5}$.
The x-coordinate of Q is $\frac{50}{n}$.
The x-coordinate of R is $\frac{72}{5+n}$.
If $PR : RQ = 3 : 5$, compute n.



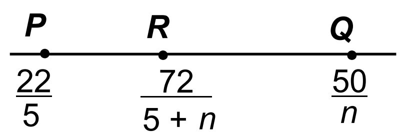
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018 SOLUTION KEY

Round 4

A) Given: $\frac{4}{25}, \frac{3}{20}, -\frac{3}{4}, \frac{7}{40}, \frac{7}{10}$
The least common denominator is 200. Converting to equivalent fraction with a common denominator, the numerators would be $32, 30, -150, 35, 140$. Thus, the first fraction will be in the middle of the list. $\frac{4}{25} = \frac{16}{100} = \underline{\mathbf{0.16}}$. (The lead zero is not required.)

B) The original perimeter was 20 units. A 40% increase adds 8 units to the perimeter.
Thus, $2(x+x) = 8 \Rightarrow x = 2$. The area of the new rectangle is 6 x $8 = 48$.
This is an increase of 24 square units, which is double the original area or a **<u>100</u>**% increase.

C) $PR : RQ = 3 : 5$



$$\Leftrightarrow \frac{\frac{72}{5+n} - \frac{22}{5}}{\frac{50}{n} - \frac{72}{5+n}} = \frac{3}{5}$$

$$\Leftrightarrow \frac{\frac{360 - 22(5+n)}{5\cancel{(5+n)}}}{\frac{50(5+n) - 72n}{n\cancel{(5+n)}}} = \frac{n(250-22n)}{5(250-22n)} = \frac{3}{5}$$

$\Rightarrow n = \underline{\mathbf{3}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) How many nonzero integer values of x over the *closed* interval $[-10,10]$ satisfy $|x+5|>2$?

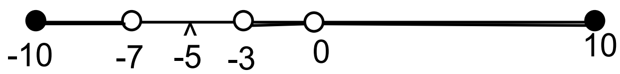
B) Let $N=\underline{T}\,\underline{U}$ be a two-digit positive integer, where $|T-U|\le 2$.
Compute the number of values of N that are prime.

C) Solve over the reals. $\big|\,|2x+5|-3\,\big|\le 1$
Your solution must not include any overlapping intervals.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018 SOLUTION KEY

Round 5

A) Since $|x+5|>2$, there are 21 integer values in the given closed interval $[-10,10]$.



All solutions must be more than 2 units from -5.
This eliminates $x=-7,-6,-5,-4,-3$, as well as $x=0$, six values.
Thus, there are **<u>15</u>** nonzero integer solutions.

Alternate solution:

$$|x+5|>2 \Leftrightarrow \begin{cases} x+5<-2 \\ x+5>+2 \end{cases} \Leftrightarrow \begin{cases} x<-7 \\ x>-3 \end{cases} \Leftrightarrow \begin{cases} -10, -9, -8 \\ -2, -1, 1, 2, 3, \ldots, 10 \end{cases}$$

B) $|T-U|=0,1,2$

Case 1: $|T-U|=0 \Rightarrow$ 11, 22, …, 99 (all multiples of 11) $\Rightarrow$ 11 only
Case 2: $|T-U|=1 \Rightarrow$ 12, 23, 34, …, 89 (or vice versa) $\Rightarrow$ 23, 43, 67, 89
Case 3: $|T-U|=2 \Rightarrow$ 13, 24, 35, …, 79 (or vice versa) $\Rightarrow$ 13, 31, 53, 79, 97
Thus, a total of **<u>10</u>** prime values.

C) $||2x+5|-3|\le 1 \Leftrightarrow -1\le |2x+5|-3\le 1 \Leftrightarrow 2\le |2x+5|\le 4$

$\Leftrightarrow |2x+5|\ge 2$ and $|2x+5|\le 4$

$\Leftrightarrow (2x+5\le -2$ or $2x+5\ge 2)$ and $(-4\le 2x+5\le 4)$

$\Leftrightarrow \left(x\le -\frac{7}{2} \text{ or } x\ge -\frac{3}{2}\right)$ and $\left(-\frac{9}{2}\le x\le -\frac{1}{2}\right)$

Taking the intersection, we have $\underline{\boldsymbol{-\frac{9}{2}\le x\le -\frac{7}{2} \text{ or } -\frac{3}{2}\le x\le -\frac{1}{2}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A) _______________________________

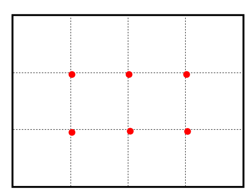
B) _______________________________

C) _______________________________

A) One mile equals 5280 feet.
A high school track star (and Olympic hopeful) ran 600 yards in 75 seconds.
To the nearest integer, what was his speed in miles per hour?

B) Let A be the 4^{th} multiple of 3 greater than 10.
Let B be the 5^{th} multiple of 4 greater than A.
Let C be the 8^{th} multiple of 6 greater than B.
Compute C.

C) A screen for sifting loam (i.e., dirt) consists of ¼" square holes.
A ¾" x 1" screen shows six interior intersection points as indicated in the diagram at the right. My sifter screen is 4 feet 6 inches by 6 feet 2 inches.
Compute the number of interior intersection points on this screen.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018 SOLUTION KEY

Round 6

A) $\dfrac{600\,\cancel{\text{yards}}}{75\,\cancel{\text{sec}}}\cdot\dfrac{60\,\cancel{\text{sec}}}{1\,\cancel{\text{min}}}\cdot\dfrac{60\,\cancel{\text{min}}}{1\,\text{hour}}\cdot\dfrac{3\,\cancel{\text{feet}}}{1\,\cancel{\text{yard}}}\cdot\dfrac{1\,\text{mile}}{5280\,\cancel{\text{feet}}}=\dfrac{600(60)^2 3}{75(5280)}=\dfrac{8(60)^2 3}{5280}=\dfrac{(60)^2 3}{660}=\dfrac{60^2}{220}=\dfrac{180}{11}=16\dfrac{4}{11}$

$\Rightarrow$ **<u>16</u>** mi/hr

<u>FYI</u>: The world record in the 400 meters (about 8 feet short of the ¼ mile) was set by Ben in 1995 at 43.18 seconds, a speed of approximately 20.75 mi/hr.

B) Brute Force:

$3k>10$		$4k>21$		$6k>40$
12 15 18 $A=21$	|	24 28 32 36 $B=40$	|	42 48 54 60 66 72 78 **<u>84</u>**

A shortcut: $\begin{cases} A=12+3(3)=21 \\ B=24+4(4)=40 \\ C=42+7(6)=\underline{\mathbf{84}} \end{cases}$

C) The screen will show $(6\cdot 12+2)4=296$ squares horizontally and $(4\cdot 12+6)4=216$ squares vertically.
Thus, the interior of the large rectangle is subdivided by 295 vertical lines and 215 horizontal lines. Each horizontal intersects each vertical so there are $295\cdot 215=\underline{\mathbf{63{,}425}}$.
Was there an alternative to brute force multiplication?
Think of $295\cdot 215$ as the product of a sum and a difference:
$(255+40)\cdot(255-40)=255^2-40^2$
Think of 255^2 as $(250+5)^2=62500+2500+25=65025$, or,
Using the shortcut squaring-a-number-ending-in-5 technique.
$255^2=100(25\cdot 26)+25=65000+25=65025$
Thus, the required difference is $65025-1600=\underline{\mathbf{63{,}425}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018
ROUND 7 TEAM QUESTIONS
ANSWERS

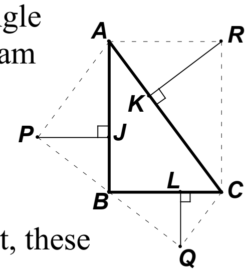
A) ____________ : ____________ D) __________________________

B) __________________________ E) __________________________

C) (__________ , __________) F) __________________________

A) In a rectangular solid with dimensions 3 x 4 x 6, a plane is parallel to opposite faces with the smallest area, and is perpendicular to the other two pairs of opposite faces. If the ratio of the total surface areas of the two rectangular solids formed by this plane is 13 : 20, compute the ratio of their volumes.

B) Right triangles with sides in a 3:4:5 ratio are drawn on each side of a right triangle whose sides have lengths $BC = 3$, $AB = 4$, and $AC = 5$, as indicated in the diagram at the right. $PB < PA < AB$, $QC < QB < BC$ and $RA < RC < AC$. Altitudes are drawn from points P, Q, and R. Compute $PJ + QL + RK$.



C) The line $(y-2) = m(x+1)$ intersects the line $Ax + By = 9$ at $(3,5)$. If, in fact, these equations define the same line, compute the ordered pair (A,B).

D) Let A and B be integers where $1 < A < 500$ and $1 < B < 600$. Compute the number of distinct ordered pairs of integers (A,B) for which $N = \left(23 - \frac{A}{7}\right)\left(41 - \frac{B}{11}\right)$ is the positive integral product of two non-integers.

E) Compute the area of the region bounded by $y = \frac{x}{5} + 8$ and $y = |x+8| - |x+7| + |x+3|$.

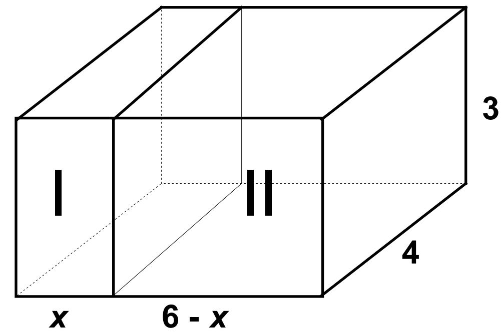
F) A Numbrix puzzle (Marilyn vos Savant) is a 9 x 9 grid which must be filled with the numbers from 1 to 81 so that consecutive numbers follow a horizontal or vertical path – no diagonals!

Compute the largest prime factor of the sum of the numbers in the highlighted cells of the completed puzzle.

11		9		5		75		77
15								63
27								61
29								59
31		33		47		49		51

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018 SOLUTION KEY

Team Round



A) The surface area of solid #1 is $2(12+4x+3x)$.

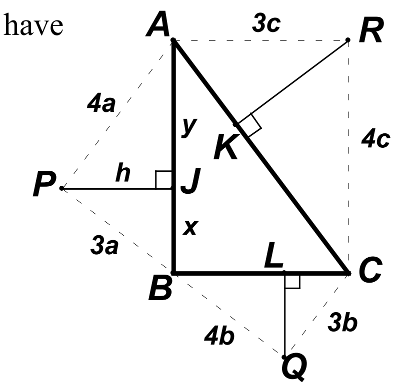
The surface area of solid #2 is $2(12+4(6-x)+3(6-x))$.

Thus, $\dfrac{24+14x}{108-14x}=\dfrac{12+7x}{54-7x}=\dfrac{13}{20}\Rightarrow$

$240+140x=702-91x\Rightarrow x=\dfrac{702-240}{140+91}=\dfrac{462}{231}=2$

and the ratio of the smaller to the larger volume is $\dfrac{2\cdot\cancel{3\cdot4}}{4\cdot\cancel{3\cdot4}}=\underline{\dfrac{\mathbf{1}}{\mathbf{2}}}$.

B) Let $(BC,AB,AC)=(3,4,5)$.



Applying the Pythagorean Theorem to ΔAPB, ΔBQC, and, ΔCRA, we have $25a^2=16$, $25b^2=9$, and $25c^2=25$. Equating area expressions, we have

$\dfrac{1}{2}\cdot3a\cdot4a=\dfrac{1}{2}\cdot12a^2=\dfrac{1}{2}\cdot h\cdot4\Rightarrow12\left(\dfrac{16}{25}\right)=4h\Rightarrow h=PJ=\dfrac{48}{25}$.

Similarly, in ΔBQC, $12b^2=3QL\Rightarrow QL=\dfrac{36}{25}$;

in ΔARC, $12c^2=5RK\Rightarrow RK=\dfrac{12}{5}$.

Thus, $PJ+QL+RK=\dfrac{48}{25}+\dfrac{36}{25}+\dfrac{12}{5}=\dfrac{48+36+60}{25}=\underline{\dfrac{\mathbf{144}}{\mathbf{25}}}$ (or **<u>5.76</u>**).

Challenge: Is $\angle KJL$ a right angle? Is ΔKJL a 3-4-5 right triangle? Prove your contentions.

C) For a point of intersection at $(3,5)$, we have $\begin{cases}(5-2)=m(3+1)\\3A+5B=9\end{cases}$.

Thus, $m=\dfrac{3}{4}$. Since parallel lines have equal slopes, and $Ax+By=9$, in slope-intercept form is $y=-\dfrac{A}{B}x+\dfrac{9}{B}$, we have $-\dfrac{A}{B}=\dfrac{3}{4}\Leftrightarrow4A+3B=0$ or $A=-\dfrac{3}{4}B$.

$\begin{cases}4(3A+5B=9)\\3(4A+3B=0)\end{cases}\Rightarrow\begin{cases}12A+20B=36\\12A+9B=0\end{cases}\Rightarrow11B=36\Rightarrow(A,B)=\underline{\left(-\dfrac{\mathbf{27}}{\mathbf{11}},\dfrac{\mathbf{36}}{\mathbf{11}}\right)}$.

Check: $3\left(-\dfrac{27}{11}\right)+5\left(\dfrac{36}{11}\right)=\dfrac{180-81}{11}=\dfrac{99}{11}=9$ and $4\left(-\dfrac{27}{11}\right)+3\left(\dfrac{36}{11}\right)=\dfrac{108-108}{11}=0$

Alternately, substituting (3, 5) in the first equation, $m=3/4$. The equation of the second line must be $3x-4y=C$, where $C=3(3)-4(5)=-11$, but C should be + 9.

Multiplying by –9/11 gives $-\dfrac{27}{11}x+\dfrac{36}{11}y=9\Rightarrow(A,B)=\underline{\left(-\dfrac{\mathbf{27}}{\mathbf{11}},\dfrac{\mathbf{36}}{\mathbf{11}}\right)}$.

Team Round - continued

D) Suppose both $\left(23-\frac{A}{7}\right)=\frac{161-A}{7}$ and $\left(41-\frac{B}{11}\right)=\frac{451-B}{11}$ are non-integers. Since 7 and 11 are relatively prime, the only way the product of these two expressions could produce a positive integer is if $161-A$ were a multiple of 11 (***but not 7***), and $451-B$ were a multiple of 7 (***but not 11***).

[For integers k and j, $\frac{161-A}{7}\cdot\frac{451-B}{11}=\frac{11k}{7}\cdot\frac{7j}{11}=kj$, an integer.]

Select A so that $\frac{161-A}{7}=\frac{11k}{7}$ *is a reduced fraction, i.e., 11k is not a multiple of 7.*

$(161-A)$ is a *positive* multiple of 11 for $A=7, 18, 29, ..., 150$, namely, 154, 143, 132, ..., 11.

These A-values may be written as $7=7+11\cdot 0$, ..., $150=7+11\cdot 13$. Thus, there are 14 such values, but $A=7$ produces 154 which is also a multiple of 7. Since numbers which are consecutive multiples of both 7 and 11 differ by 77, we must also exclude $A=84$.
Therefore, we have irreducible positive fractions for 14 – 2 = $\boxed{12}$ different A-values.

$(161-A)$ is a *negative* multiple of 11 for $A=172, ..., 491$, namely, -11, ..., -330.

These A-values may be written as $172=7+11\cdot 15$, ..., $491=7+11\cdot 44$. Thus, there are 44 – 15 + 1 = 30 such values. Repeatedly adding 77 to the previous "troublesome" value (of 84), $A=\cancel{161}$, 238, 315, 392, 469 produce negative multiples of 7 which must be excluded.
Therefore, we have irreducible negative fractions for 30 - 4 = $\boxed{26}$ different A-values.

Select B so that $\frac{451-B}{11}=\frac{7j}{11}$ *is a reduced fraction, i.e., 7j is not a multiple of 11.*

$(451-B)$ is a *positive* multiple of 7 for $B=3$ ..., 444, namely, 448, ..., 7.

These B-values may be written as $3=3+7\cdot 0$, ..., $444=3+7\cdot 63$. Thus, there are 64 such values, but $B=66$ produces 385 which is also a multiple of 11. Repeatedly adding 77 to this "troublesome" value, $B=143, 220, 297, 374$ also produce positive multiples of 7 which must be excluded. We have irreducible positive fractions for 64 – 5 = $\boxed{59}$ B-values.

$(451-B)$ is a *negative* multiple of 7 for $B=458, ..., 598$, namely, -7, ..., -147.

These B-values may be written as $458=3+7\cdot 65$, ..., $598=3+7\cdot 85$

Thus, there are 21 such values. Adding 77 (to the previous value of 374), $B=\cancel{451}$, 528 produces -77, a negative multiple of 11 which must be excluded. We have irreducible negative fractions for 21 – 1 = $\boxed{20}$ B-values.

Therefore, $12\cdot 59+26\cdot 20=708+520=$ **<u>1228</u>** ordered pairs (A,B) produce a positive integer for the product of two non-integers $N=\left(23-\frac{A}{7}\right)\left(41-\frac{B}{11}\right)$.

Team Round - continued

E) Given: $y = \frac{x}{5} + 8$ and $\boxed{y = |x+8| - |x+7| + |x+3|}$

Both functions are continuous, the graph of the former being a straight line, and the latter being the union of two segments and two rays. The critical points are at $x = -8$, $x = -7$, and $x = -3$.

Thus, in the sketch below, the coordinates of A, B and C are easily determined to be $A(-8,4), B(-7,5), C(-3,1)$

Since $m_{AB} = +1$ and $m_{BC} = -1$, it is clear that ΔPQR is a right triangle and $ABCR$ is a rectangle. Next equivalent functions must be found for the boxed absolute value function when $x < -8$ (left of the vertical line $x = -8$, where point P is located) and $x > -3$ (right of the vertical line $x = -3$, where point Q is located).

For $x < -8$, $y = |x+8| - |x+7| + |x+3| \Leftrightarrow y = (-x-8) - (-x-7) + (-x-3) = -x-4$

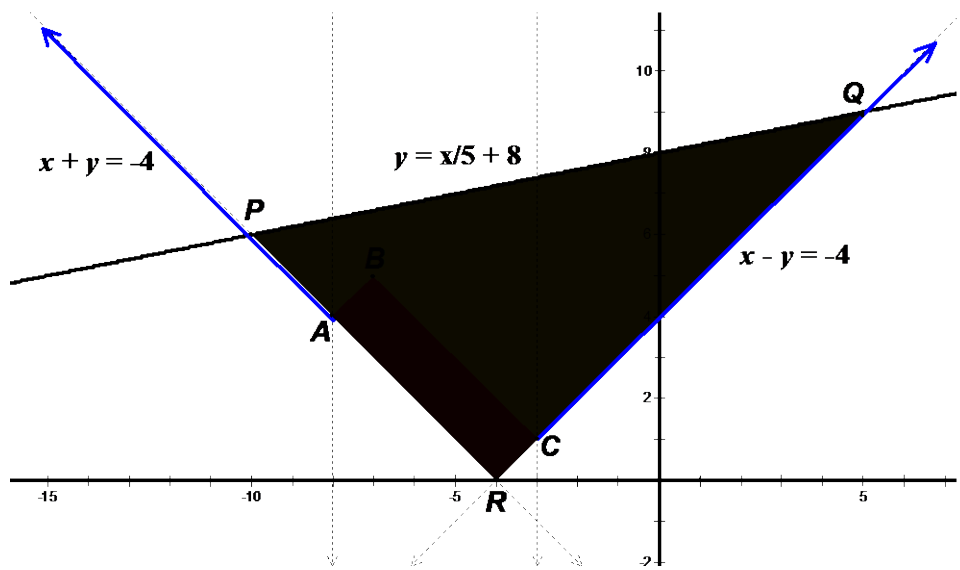
Thus, P lies on the line $x + y = -4$. Similarly, Q lies on the line $x - y = -4$.

By finding the intersection of each of these lines with $y = \frac{x}{5} + 8$, we have the required coordinates $P(-10,6), Q(5,9)$. R must be located at $(-4,0)$.

$\Rightarrow PR = 6\sqrt{2},\ RQ = 9\sqrt{2} \Rightarrow \text{Area}(\Delta PQR) = 54$

$\Rightarrow AB = \sqrt{2}, BC = 3\sqrt{2} \Rightarrow \text{Area}(ABCR) = 6$.

Therefore, the required area is **<u>48</u>**.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018 SOLUTION KEY

Team Round - continued

F) Work the top and bottom rows, the leftmost and rightmost columns first filling in these outermost cells, where there is only one choice and then working inward wherever the choice is certain. This is the case for all cells outside the dark perimeter "fence". It is left to you to fill in these outside values in the grid to the right. After this has been done, there are at least two possibilities for each of the remaining loose end numbers 3, 26, 33, and 47.

11		**9**		**5**		**75**		**77**
15				3				**63**
27	26							**61**
29								**59**
31		**33**		**47**		**49**		**51**

11	10	**9**	6	**5**	74	**75**	76	**77**
12	13	8	7	4	73	80	79	78
15	14	1	2	3	72	81	64	**63**
16	17	18	39	40	71	70	65	62
27	26	19	38	41	42	69	66	**61**
28	25	20	37	44	43	68	67	60
29	24	21	36	45	56	57	58	**59**
30	23	22	35	46	55	54	53	52
31	32	**33**	34	**47**	48	**49**	50	**51**

Unless 1 and 2 are placed directly to the left of 3, a dead-end is created and the path would have to terminate in this "cul-de-sac", but that is impossible, as 1 would have already been placed (and 81 has been placed as an outside choice certain value).
25 must be placed below 26, otherwise, the connection to 17 would be impossible. Let R, L, U, and, D denote right, left, up, and, down, respectively. The path from 26 to 17 is shown in blue (light shading - D3R1U4). The path from 33 to 47 is shown in red (dark shading - R1U5R1D1R1D1L1D2).

Thus, the sum of the underscored values is 34 + 46 + 42 + 81 = 203 = 7(29).
The largest prime factor is **29**.
Are you convinced that this solution is unique?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2018 ANSWERS

Round 1 Geometry Volumes and Surfaces

A) $\frac{81\pi}{25}$ (or 3.24π) B) 12 C) 49.3 (lbs)

Round 2 Pythagorean Relations

A) 19 B) $9-\sqrt{13}$ C) 10

Round 3 Linear Equations

A) 460 B) 924 C) 12.25

Round 4 Fraction & Mixed numbers

A) 0.16 B) 100 C) 3

Round 5 Absolute value & Inequalities

A) 15 B) 10 C) $-\frac{9}{2} \le x \le -\frac{7}{2}$ or $-\frac{3}{2} \le x \le -\frac{1}{2}$

Round 6 Evaluations

A) 16 B) 84 C) 63,425

Team Round

A) 1 : 2 D) 1228

B) $\frac{144}{25}$ (or 5.76) E) 48

C) $\left(-\frac{27}{11}, \frac{36}{11}\right)$ F) 29 [46 + 34 + 42 + 81 = 203 = 29(7)]