MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2018
ROUND 1 ALGEBRA 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) ______________________________

B) ______________________________

C) (_____ , _____ , _____ , _____)

A) Given: $\begin{cases} y = 3x + c \\ x + 2y = c - 1 \end{cases}$

Compute the <u>smallest</u> positive integer value of c for which both x and y are integers.

B) Given: $\begin{vmatrix} 20 & 3.5 & A & \sqrt{6} \\ 24 & -12 & 15 & 21 \\ -4 & 7 & 2 & 6 \\ -8 & y & x & -7 \end{vmatrix} = 0$

Solve for w: $$w\begin{vmatrix} 2 & x & 1 \\ 0 & 5 & 7 \\ 2 & y & 1 \end{vmatrix} + 6w = 40$$

C) We are assured in Euclidean Geometry that "Three non-collinear points determine exactly one plane.". Assume that a plane is defined by an equation of the form $Ax + By + Cz = N$. Compute the ordered quadruple (A, B, C, N), where $A > 0$ and A, B, C and N are relatively prime integers, for the plane that contains the points $P(1,\ 0,\ 2)$, $Q(3,-1,-1)$, and $R(-2,3,7)$.

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Round 1

A) Substituting for y in the second equation, $x+2(3x+c)=c-1 \Rightarrow x=\dfrac{-1-c}{7}$

$c=\underline{\mathbf{6}} \Rightarrow (x,y)=(-1,3)$.

B) For the given 4 x 4 determinant $\begin{vmatrix} 20 & 3.5 & A & \sqrt{6} \\ 24 & -12 & 15 & 21 \\ -4 & 7 & 2 & 6 \\ -8 & y & x & -7 \end{vmatrix}$ to be zero, two rows (or 2 columns) must be dependent, i.e. one must be a multiple of the other. Comparing rows 2 and 4, we have row 2 equals row 4 times -3 and, consequently, $(x,y)=(-5,4)$.

Expanding by column 1,

$$\begin{vmatrix} 2 & -5 & 1 \\ 0 & 5 & 7 \\ 2 & 4 & 1 \end{vmatrix} = 2\begin{vmatrix} 5 & 7 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} -5 & 1 \\ 5 & 7 \end{vmatrix} = 2(5-28)+2(-35-5)=2(-23-40)=-126$$

Therefore, $-120w=40 \Rightarrow w=\underline{\mathbf{-\frac{1}{3}}}$.

C) $P(1,0,2) \Rightarrow (1)\ A+2C=N$
$Q(3,-1,-1) \Rightarrow (2)\ 3A-B-C=N$
$R(-2,3,7) \Rightarrow (3)\ -2A+3B+7C=N$

$$\left.\begin{array}{l} 2(2)+(1) \Rightarrow 7A-2B=3N \\ 7(2)+(3) \Rightarrow 19A-4B=8N \end{array}\right\} \Rightarrow 5A=2N \Rightarrow A=\frac{2}{5}N .$$

$$(1) \Rightarrow C=\frac{N-A}{2}=\frac{N-\frac{2}{5}N}{2}=\frac{3}{10}N ,$$

$$(2) \Rightarrow B=3A-C-N \Rightarrow B=\frac{6}{5}N-\frac{3}{10}N-N=-\frac{1}{10}N$$

Thus, $(A,B,C,N)=\left(\frac{2}{5}N,-\frac{1}{10}N,\frac{3}{10}N,N\right)$.

Taking N to be the least common multiple of 5 and 10, we have $(A,B,C,N)=\underline{\mathbf{(4,-1,3,10)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2018
ROUND 2 ALGEBRA 1: EXPONENTS AND RADICALS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Compute <u>all</u> real values of x for which $4^x - 17\left(2^x\right) + 2^4 = 0$.

B) $64^{\frac{P}{Q}}$ has an integer value strictly between 1 and 64, where $\frac{P}{Q}$ is a reduced proper fraction.
Let $S = P + Q$. Compute the <u>sum</u> of all possible distinct values of S.

C) If both sides of the equation $\sqrt{11-7x} - \sqrt{5-2x} = 14$ are squared, two solutions over the real numbers are obtained for this *derived* quadratic equation. One solution is extraneous and one solves the original equation. If the extraneous solution is $x = -10$, determine the other solution and express it as a decimal accurate to the nearest hundredth.

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Round 2

A) Let $y = 2^x \Leftrightarrow y^2 = \left(2^x\right)^2 = \left(2^2\right)^x = 4^x$

$4^x - 17\left(2^x\right) + 2^4 = 0 \Leftrightarrow y^2 - 17y + 16 = \left(y-16\right)\left(y-1\right) = 0$

$y = 2^x = 1, 16 \Rightarrow x =$ **<u>0, 4</u>**.

The following should also be accepted: 0 or 4, 0 and 4.

B) Since $64^{\frac{1}{6}} = \left(2^6\right)^{\frac{1}{6}} = 2^1 = 2$, the largest possible integer value of Q is 6.

P can be any integer value from 1 through 5 and $\frac{P}{Q} = \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}$, resulting in

S = 7, 4, 3, 5 and 11. The required sum is **<u>30</u>**.

C) Given: $\sqrt{11-7x} - \sqrt{5-2x} = 14$

Squaring both sides, $11 - 7x - 2\sqrt{11-7x} \cdot \sqrt{5-2x} + 5 - 2x = 196$

$\Leftrightarrow 16 - 9x - 2\sqrt{11-7x} \cdot \sqrt{5-2x} = 196 \Leftrightarrow -2\sqrt{11-7x} \cdot \sqrt{5-2x} = 9x + 180 = 9\left(x+20\right)$

Squaring both sides again, $4\left(11-7x\right)\left(5-2x\right) = 81\left(x+20\right)^2$

$\Leftrightarrow 220 - 228x + 56x^2 = 81x^2 + 3240x + 32400$

$\Leftrightarrow 25x^2 + 3468x + 32180 = 0$

Since we know $x = -10$ is the extraneous solution, factoring the trinomial is easy.

It must be $\left(x+10\right)\left(25x+3218\right)$. Setting the second factor equal to zero, we have

$x = -\frac{3218}{25} = -\frac{4 \cdot 3218}{4 \cdot 25} = -\frac{12872}{100} =$ **<u>-128.72</u>** (exactly).

Notice:

Multiplying by the single digit 4 is always easier than dividing by the two-digit number 25.

Check: $\sqrt{11+7\left(128.72\right)} = \sqrt{912.04} = 30.2$ (exactly)

$\sqrt{5+2\left(128.72\right)} = \sqrt{262.44} = 16.2$ (exactly) $\Rightarrow$ difference: 14

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2018
ROUND 3 TRIGONOMETRY: ANYTHING

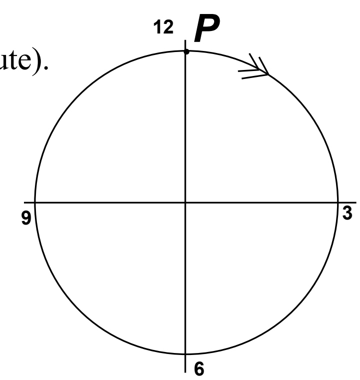
ANSWERS

A) ________________________________

B) ________________________________

C) (____ , ____ , ____________)

A) Initially, point P on the shaft of a motor is at the 12:00 position.
The shaft is spinning clockwise at 10,000 RPM (revolutions per minute).
In one second, how many times does point P pass the 9:00 position?



B) $z = x + yi$, where $i = \sqrt{-1}$, is the solution of the following equation

$$|30 - 72i|z - (2\cos\pi)z = |-21 + 72i| - \left(5\tan\left(-\frac{7\pi}{4}\right)\right)z$$

Compute <u>all</u> possible ordered pairs (x, y).

C) In trigonometric form, $(-16 + 30i)^{50}$ is equivalent to $A^B cis(B\theta)$. Compute the ordered triple (A, B, θ), where θ is in <u>radians</u> and both A and B are positive. If necessary, express θ in terms of $Tan^{-1}(x)$, where $x > 0$.

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Round 3

A) $\dfrac{10,000}{60} = 166.\overline{6} = 166\dfrac{2}{3}$ rev / sec . Since $\dfrac{2}{3} < \dfrac{3}{4}$, P has passed the 9:00 position only <u>**166**</u> times.

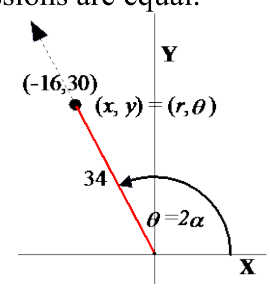
B) The absolute value of the complex number $a + bi$ is $\sqrt{a^2 + b^2}$.
Trying to avoid tedious arithmetic, let's determine whether (30, 72, n) is a Pythagorean triple.
(30, 72, n) = 6(5, 12, ?) $\Rightarrow$? = 13 and n = 78 $\Rightarrow$ $|30 - 72i|$ = 78
Similarly, $|-21 + 72i|$ = 75. [(21, 72, 75) = 3(7, 24, 25)]
Thus, the intimidating equation simplifies to $78z + 2z = 75 - 5z$ or $85z = 75$
$\Rightarrow z = \dfrac{15}{17} = \dfrac{15}{17} + 0i \Rightarrow (x, y) = \underline{\left(\mathbf{\dfrac{15}{17}, 0}\right)}$.

C) Converting $(-16 + 30i)$ to trigonometric form, we notice the real and imaginary parts of the complex number are twice the legs of the Pythagorean Triple (8, 15, 17), so r = 34 and θ is an angle in the second quadrant whose tangent has a value of –15/8. $Tan^{-1}\left(\dfrac{15}{8}\right)$ is in quadrant 1, so $\theta = \pi - Tan^{-1}\left(\dfrac{15}{8}\right)$.

Thus, $(rcis\theta)^{50} = r^{50}cis(50\theta) \Rightarrow (A, B, \theta) = \underline{\left(\mathbf{34,\ 50,\ \pi - Tan^{-1}\left(\dfrac{15}{8}\right)}\right)}$.

However, $50\theta = 50\left(\pi - \text{Tan}^{-1}\left(\dfrac{15}{8}\right)\right) = 50\pi + 50 \cdot -\text{Tan}^{-1}\left(\dfrac{15}{8}\right)$ and 50π just denotes 25 revolutions around the unit circle, so $(A, B, \theta) = \underline{\left(\mathbf{34,\ 50,\ -Tan^{-1}\left(\dfrac{15}{8}\right)}\right)}$ is also acceptable.

FYI: Interestingly, if the required equivalent expression were of the form $A^B cis(C\theta)$, where B and C could be different values, then many other equivalent expressions would have been possible. With some difficulty, it can be shown that $100\,\text{Tan}^{-1}\left(\dfrac{5}{3}\right) = 50\pi - 50\,\text{Tan}\left(\dfrac{18}{8}\right)$.
Evaluation with a calculator will give convincing evidence that the two expressions are equal.



<u>Brandon Pho (Canton)</u>: Let $\theta = 2\alpha$. Then: $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan\alpha} = -\dfrac{15}{8}$

Cross multiplying, $16\tan\alpha = -15 + 15\tan^2\alpha \Rightarrow (5\tan\alpha + 3)(3\tan\alpha - 5) = 0$

Since the argument of Tan^{-1} was required to be positive, we have $\tan\alpha = \dfrac{5}{3}$.

Therefore, $C\theta = 50(2\alpha) = 100\alpha = 100\,\text{Tan}^{-1}\left(\dfrac{5}{3}\right)$.

The original question opened a can of worms in terms of potential alternative answers, since no restrictions were put on *A*, *B*, and *C*.

In trigonometric form, $(-16+30i)^{50}$ is equivalent to $A^{B}cis(C\theta)$. Compute the ordered quadruple (A,B,C,θ), where θ is in <u>radians</u>. If necessary, express θ in terms of $Tan^{-1}(x)$, where $x > 0$.

Converting a complex number in $a+bi$ form to trigonometric form $r(\cos\theta + i\sin\theta)$ - frequently abbreviated $rcis\theta$, where *r* denotes the distance of the point from the origin and θ denotes the angle formed by the ray with an endpoint at the origin and passing through the given point and the positive *X*-axis. Note that different $rcis\theta$ values can represent the same $a+bi$ value because of coterminal values of θ. The conversion is accomplished using the Pythagorean Theorem and SOHCAHTOA.

Specifically, $\begin{cases} r^2 = a^2 + b^2 \\ \tan\theta = \dfrac{b}{a} \end{cases}$. By inspection, we know what quadrant $a+bi$ is in, so we know in what quadrant the terminal side of the angle must be in. We chose θ accordingly.

Secondly, by DeMoivre's theorem we know $(rcis\theta)^n = r^n \cdot cis(n\theta)$. That is, the original radius is raised to the n^{th} power and the original angle is multiplied by *n*.

First base: $(-16+30i) = 34cis\theta$, where $\tan\theta = -\dfrac{30}{16} = -\dfrac{15}{8}$

Second base - Representing θ:

$\mathrm{Tan}^{-1}(x)$ denotes the principal inverse tangent. It always denotes an angle in either Q1 or Q4.

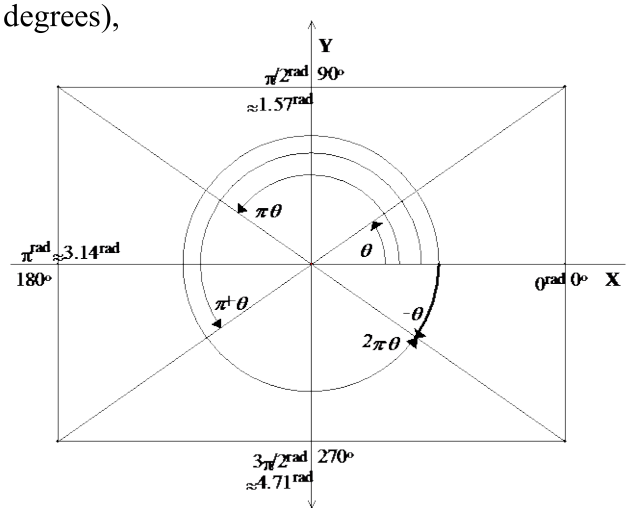
Specifically, $y = \mathrm{Tan}^{-1}(x) \Leftrightarrow x = \tan(y)$, where the following conventions are adopted.

For $x \geq 0$, $0 \leq y < \dfrac{\pi}{2}$ (in radians) or $0° \leq y < 90°$ (in degrees), i.e. Quadrant 1.

If necessary, add or subtract $2\pi\ (or\ 360°)$ until *y* is in the required interval.

For $x < 0$, $-\dfrac{\pi}{2} < y < 0$ (in radians) or $-90° < y < 0°$ (in degrees), i.e. Quadrant 4.

If necessary, add or subtract $2\pi\ (or\ 360°)$ until *y* is in the required interval.



$\mathrm{Tan}^{-1}\left(\dfrac{15}{8}\right)$ denotes a Q1 angle. We require the related angle in Q2.

In radians, that's $\theta = \pi - \mathrm{Tan}^{-1}\left(\dfrac{15}{8}\right)$. [In degrees, it would be $180° - \mathrm{Tan}^{-1}\left(\dfrac{15}{8}\right)$.] By travelling around the unit circle in the clockwise direction, you might see other equivalent expressions for θ.

Third base: Raise the trig form of the complex number to the required power.

$$\left(34cis\left(\pi - \mathrm{Tan}^{-1}\left(\frac{15}{8}\right)\right)\right)^{50} = 34^{50}cis\left(50\left(\pi - \mathrm{Tan}^{-1}\left(\frac{15}{8}\right)\right)\right)$$

Homerun:
$34^{50} = (-34)^{50} \Rightarrow B = 50,\ A = \pm 50$
But what about $C\theta$?
Let's compare several possibilities, using numerical approximations.
The most straightforward answer was
$C\theta = 50\left(\pi - \mathrm{Tan}^{-1}\frac{15}{8}\right)$. Plugging into a calculator, this is approximately 103.0376827 radians.
We can subtract multiples of 2π until we get a coterminal value between 0 and 6.28 (i.e., 2π)
Subtracting 32π, we get **2.506717738** radians which is in quadrant 3.
So any equivalent answer must have a reference angle coterminal with this value.

Let's check out $-50\,\mathrm{Tan}^{-1}\left(\frac{15}{8}\right) \approx -54.04195003$ radians.
Adding 18π, we get the bolded quantity above.

Checking the least likely candidate (The origin of this answer was included in the comments with the original solution to 3C. Thank you – Brandon):

$100\,\mathrm{Tan}^{-1}\left(\frac{5}{3}\right) \approx 103.0376827$ which was our first approximation
If you feel really motivated perhaps you'd like to tackle establishing the following equality:

$100\,\mathrm{Tan}^{-1}\left(\frac{5}{3}\right) - 50\,\mathrm{Tan}^{-1}\left(\frac{15}{8}\right) = 50\pi$ (without appealing to approximations!).

Another possibility might be $-100\,\mathrm{Tan}^{-1}\left(\frac{3}{5}\right) \approx -54.04195003$ and again we have equivalency.
If we had represented θ as 3α (instead of 2α) and we happened to know the expansion for $\tan(3\alpha)$ and felt ambitious we might find three more equivalent answers, if the resulting cubic equation has 3 real roots. Dare I suggest $\theta = n\alpha$ might give us more equivalent answers as well as a serious migraine. I trust no one took this route, but I love to see what further investigation might turn up, just not tonight.
This question has forced all of us to take a much closer look to the representation of complex numbers in trigonometric form. Can we call this a fortunate mistake?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2018
ROUND 4 ALGEBRA 1: ANYTHING

ANSWERS

A) ______________________________

B) _________________________ hours

C) ______________________________

A) I'm thinking of a number.
Subtract 1 from the number and the resulting number (which is also positive) is then divisible by each of the nonzero <u>digits</u> (in base 10). What is the <u>smallest</u> positive integer with this property?

B) Let $\mathcal{L}$, a line through the center of a sphere, be an axis of rotation.
A sphere turns through 15° in 1 hour, making a complete revolution in 24 hours.
This sphere models Earth, where 1 day is equivalent to 24 hours.
How long (to the nearest 0.1 hour) is a day on Saturn, if Saturn turns through 150% <u>more</u> degrees than Earth does in one hour?

C) The inequality $k \le |2x-1| \le 11$, where k is an integer, has exactly 10 integer solutions.
Compute all possible values of k.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2018 SOLUTION KEY

Round 4

A) If the original number were 1, then, after subtracting 1, the resulting number would be 0, which is divisible by all numbers (except 0), but this possibility is eliminated, since both numbers had to be positive.
The required number is 1 more than the least common multiple of the digits from 2 to 9. (1 is ignored, since every integer is divisible by 1).
Looking at the prime factorization of these digits $\left(2,\ 3, 2^2,\ 5,\ 2\cdot 3,\ 7,\ 2^3,\ 3^2\right)$, the only primes occurring are 2, 3, 5 and 7. The smallest integer divisible by all of the digits is $2^3\cdot 3^2\cdot 5\cdot 7 = 2520$. Adding 1, we have **<u>2521</u>**, the number I was thinking of.

B) Saturn turns $15+15\cdot\frac{3}{2}=37\frac{1}{2}^{\circ}$ in 1 hour.
Thus, a full revolution takes $\frac{360}{37.5}=\frac{720}{75}=\frac{48}{5}=\underline{\mathbf{9.6}}$ hours.
FYI: The currently accepted length of a "day" on Saturn is 10.2 hours – the time it takes for a complete revolution around its axis.

C) $k \le |2x-1| \le 11 \Rightarrow |2x-1| \ge k$ and $|2x-1| \le 11$
$|2x-1| \ge k \ \Rightarrow\ 2x-1 \le -k$ or $2x-1 \ge +k \ \Rightarrow\ x \le \frac{1-k}{2}$ or $x \ge \frac{1+k}{2}$ (inner limits)
$|2x-1| \le 11 \ \Rightarrow\ -11 \le 2x-1 \le +11 \ \Rightarrow\ -5 \le x \le 6$ (outer limits)

Therefore, the solution in general consists of two segments of length $(11 - k)/2$.
There will be the same number of solutions on both segments. Thus, there will be exactly 5 integer solutions on each closed interval.
On the right, solutions must be 2…6 which implies that
$1<\frac{1+k}{2}\le 2 \Rightarrow 2<1+k\le 4 \Rightarrow 1<k\le 3 \Rightarrow k=\underline{\mathbf{2,\ 3}}$
You should verify that for these two values of k, there are exactly 5 integer solutions.
If we started on the left side, we would require that $-1\le\frac{1-k}{2}<0$.
Show that this is true if and only if $1<k\le 3$, i.e., $k=2,3$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2018
ROUND 5 PLANE GEOMETRY: ANYTHING

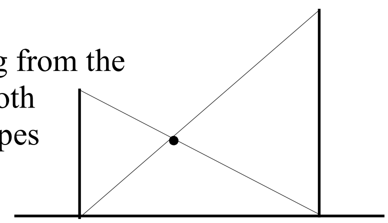
ANSWERS

A) ________________________________

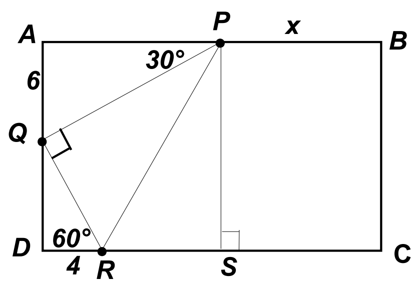
B) ________________________________

C) ________________________________

A) Two buildings of heights 60 feet and 40 feet have taut ropes running from the top of one to the bottom of the other. Assuming the buildings are both perpendicular to level ground, how high above the ground do the ropes intersect?

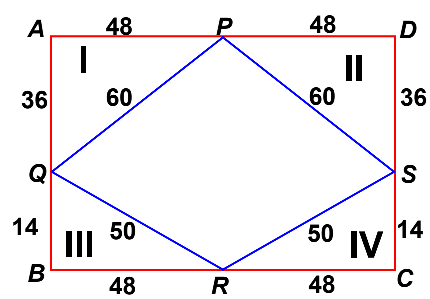


B) In rectangle *ABCD*, $\overline{PQ} \perp \overline{QR}$, $\overline{PS} \perp \overline{DC}$, $AQ = 6$, $DR = 4$, $m\angle APQ = 30°$, $m\angle DRQ = 60°$, and $PB = x$, as indicated in the diagram at the right. For a specific value of *x*, the perimeter of rectangle *PBCS* equals the perimeter of ΔPQR. Compute $\lfloor x \rfloor$.



Note: $\lfloor x \rfloor$ denote the *largest* integer which is less than or equal to *x*.

C) Two congruent *scaled* copies of the 3-4-5 right triangle (I and II) and two congruent *scaled* copies of the 7-24-25 (III and IV) right triangle are to be laid out with legs on the sides of rectangle *ABCD* so that a rhombus is formed inside the rectangle by the 4 hypotenuses. The diagram at the right is a **failed** attempt. Although *ABCD* is a rectangle, *PQRS* is <u>not</u> a rhombus.



The scale factors are both integers, both greater than 1, but not necessarily equal. If *ABCD* must have a <u>minimum</u> area, compute the area of the rhombus *PQRS*.

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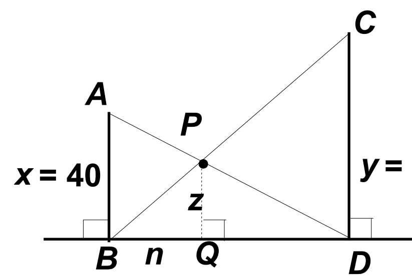
Round 5



A) The distance between the bases of the buildings is irrelevant.
So, let $BD=1$.

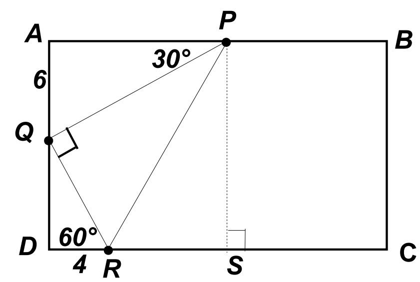
$\Delta BPQ \sim \Delta BCD \Rightarrow \frac{z}{60}=\frac{n}{1}$.

$\Delta DPQ \sim \Delta DAB \Rightarrow \frac{z}{40}=\frac{1-n}{1} \Leftrightarrow n=1-\frac{z}{40}$

Thus, $\frac{z}{60}=1-\frac{z}{40} \Leftrightarrow 2z=120-3z \Rightarrow z=\frac{120}{5}=\underline{\mathbf{24}}$.

In general, it is easy to show that if $\overline{PQ} \parallel \overline{AB} \parallel \overline{CD}$, $\frac{1}{z}=\frac{1}{x}+\frac{1}{y}$.

$\frac{1}{z}=\frac{1}{60}+\frac{1}{40} \Rightarrow \frac{1}{z}=\frac{40+60}{40\cdot 60}=\frac{1\cancel{00}}{4\cancel{0}\cdot 6\cancel{0}} \Rightarrow z=\underline{\mathbf{24}}$.

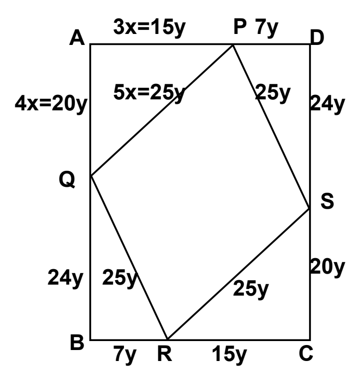


B) In ΔAPQ, $PQ = 12$ and $AP = 6\sqrt{3}$.
In ΔDRQ, $QR = 8$ and $DQ = 4\sqrt{3}$.
In ΔPQR, $PR=\sqrt{12^2+8^2}=\sqrt{208}=\sqrt{16\cdot 13}=4\sqrt{13}$
Thus, perimeter of ΔPQR is $20+4\sqrt{13}$ and

$20+4\sqrt{13}=2\left(6+4\sqrt{3}\right)+2x \Rightarrow$

$x=10+2\sqrt{13}-6-4\sqrt{3}=4+2\left(\sqrt{13}-2\sqrt{3}\right)=4+2\left(\sqrt{13}-\sqrt{12}\right)$.

Clearly, the quantity in parentheses is between 0 and 1 and much closer to 0.
Thus, $\lfloor x \rfloor = 4+0=\underline{\mathbf{4}}$.



C) In the arrangement to the right, let x and y denote the scale factors.
To insure a *PQRS* is a rhombus, we require $5x = 25y \Rightarrow x = 5y$
Note that with a fixed perimeter, as a rectangle becomes more like a square, its area increases; and, conversely, as the rectangle becomes less like a square, its area decreases. Thus, we require that the *shorter* sides of each right triangle lie along the same side of the rectangle. The lengths of all segments can now be expressed in terms of y, as indicated. The minimum possible value of y is 2.
The area of *ABCD* is 44 x 88 = 3872.
The area of the triangular cutoffs is 30(40) + 14(48) = 1872
$\Rightarrow$ 3872 – 1872 = **<u>2000</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2018
ROUND 6 ALGEBRA 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) ______________________________

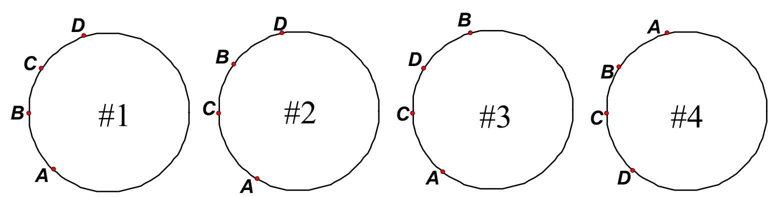
B) ______________________________

C) ______________________________

A) Small sheets of paper are numbered 1 through 15, inclusive, and placed in a hat.
Five sheets are drawn simultaneously at random.
Compute the probability that the page number of exactly two sheets will be prime.
Express your answer as a reduced fraction.

B) Compute the sum of the exponents in the two middle terms in the expansion of $\left(5M^{2x} - N^{-3x}\right)^9$.

C) <u>Aside:</u>
Consider the problem of arranging 4 charms on a Pandora bracelet.
Bracelets #1, #2 and #3 are identical, but #4 is identical to #1, if the bracelet is flipped over, i.e., if the charms on #1 are read clockwise (CW) and the charms on #4 are read counterclockwise (CCW). Thus, there are only 3 distinct arrangements of a bracelet with 4 charms. This is a variation of a circular permutation, where *flipping over is permitted*.



<u>The problem</u>:
A burglar was stealing a bracelet with 5 different charms when he inadvertently dropped it. The clasp opened, and the charms fell off. Considering this a bad omen, he put the charms back on the bracelet and returned it to the jewelry box. In how many ways could the charms have been put back on the bracelet in the <u>wrong</u> order, that is, in an order different from the original?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2018 SOLUTION KEY

Round 6

A) There are 6 primes, namely, 2, 3, 5, 7, 11 and 13, and 9 non-primes.

Five sheets can be drawn in $\binom{15}{5}$ different ways.

Two sheets with a prime number <u>and</u> 3 sheets with a non-prime number must be drawn.

This can be done in $\binom{6}{2}\cdot\binom{9}{3}$ ways.

Thus, the required probability is $\dfrac{\binom{6}{2}\cdot\binom{9}{3}}{\binom{15}{5}} = \dfrac{\frac{6\cdot 5}{1\cdot 2}\cdot\frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}}{\frac{\cancel{15}\cdot 14\cdot 13\cdot 12\cdot 11}{1\cdot 2\cdot\cancel{3}\cdot 4\cdot\cancel{5}}} = \dfrac{\cancel{15}^{5}\cdot 12\cdot\cancel{7}}{\cancel{7}\cdot 13\cdot\cancel{3}\cdot 11} = \underline{\dfrac{\mathbf{60}}{\mathbf{143}}}$.

B) There are 10 terms in the expansion. The middle terms are 5^{th} and the 6^{th}.

In general, the k^{th} term in the expansion of $(A+B)^n$ is $\binom{n}{k}A^{n-k}B^k$, but this assumes that the terms are indexed 0 to 9. Thus, we must use $k = 4$ and $k = 5$.

Substituting $n = 9$, $A = 5M^{2x}$, and $B = -N^{-3x}$

The sum of the middle terms is $\binom{9}{4}\left(5M^{2x}\right)^5\left(-N^{-3x}\right)^4 + \binom{9}{5}\left(5M^{2x}\right)^4\left(-N^{-3x}\right)^5$.

Focusing on the exponents only,

$(2x)5 + (-3x)4 + (2x)4 + (-3x)5 \Rightarrow 10x - 12x + 8x - 15x = \underline{\mathbf{-9x}}$.

C) In a straight line, n objects can be arranged in $n!$ distinct ways. However, when arranging the same n objects around a circle, any single object could be the "starting" point. Different arrangements are determined by how the remaining $(n – 1)$ objects are arranged. This can be done in $(n – 1)!$ ways. By simply flipping the charm bracelet over, a clockwise ordering becomes a counterclockwise ordering. Thus, total number of different arrangements of a 5-charm bracelet is $\dfrac{(5-1)!}{2} = \dfrac{24}{2} = 12$, one of which is correct.

Thus, there are <u>**11**</u> arrangements different from the original.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2018
ROUND 7 TEAM QUESTIONS

ANSWERS

A (_______ , _______ , ______) D) (_______ , _______ , ______)

B) ________________________ E) ________________________

C) (_______ , _______ , ______) F) __________________________

A) My new cereal will have blueberries (*B*), strawberries (*S*), and peaches (*P*).
I have three suppliers who have proposed supplying 18-pound bags of dried fruit mixes in the following ratios:
Mix #1: $B:S:P=1:2:3$ Mix #2: $B:S:P=2:3:4$ Mix #3: $B:S:P=5:6:7$
My preferred mix would be $B:S:P=3:4:5$.
Since time is short, I decide to place orders for bags of dried fruit from each of the suppliers and re-mix them to achieve my desired ratio. My order is for fifty 18-pound bags. Let (x_1, x_2, x_3) denote the number of 18-pound bags of each mix I order, where x_1, x_2, and x_3 are relatively prime. Let *k* denote the number of ordered triples (x_1, x_2, x_3) which allow me to produce my desired mix, and (M, m), the maximum and minimum values, respectively, of x_3. Compute the ordered triple (k, M, m).

B) For positive integer values of the constant *k*, suppose the inequality $\sqrt{3x}-\sqrt{3x-k}>1$ has *n* distinct integer solutions. Determine the <u>minimum</u> value of *n* for which having *n* integer solutions is impossible.

C) ΔABC has sides of integer length, a perimeter of 24, and its sides and angles satisfy the proportion $\dfrac{\cos A}{b}=\dfrac{\cos B}{a}$. There are *n* such triangles. The maximum and minimum areas of one of these triangles are *M* and *m*, respectively. Compute the ordered triple (n, M, m).

D) Let the ordered triple (o, d, h) be the solutions to the equation $2x-7=AB5$, in octal (base 8), decimal (base 10), and hexadecimal (base 16), respectively. The constant on the right-hand side of the equation is a three-digit hexadecimal integer. Compute the ordered triple (o, d, h).
Remember: In base 16, the “extra” digits *A*, *B*, *C*, *D*, *E*, and *F* denote 10 … 15, respectively.

E) The perimeter of a rectangle with integer dimensions is 238 units. If a diagonal of the rectangle also has integer length, compute all possible radii of the circle circumscribed about this rectangle.

F) Compute <u>all</u> possible constant terms in the expansion of $\left(x^m+\dfrac{1}{x^2}\right)^6$, where *m* is a positive integer.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2018 SOLUTION KEY

Team Round

A) We require that $x_1 + x_2 + x_3 = 50$.

Since $50 \cdot 18 = 900 = 3n + 4n + 5n = 12n$, $n = 75$, and we must have 225, 300 and 375 lbs. of blueberries, strawberries, and peaches, respectively, in the final mix. Mixes #1, #2 and #3 contain $(3,6,9)$, $(4,6,8)$, and $(5,6,7)$ lbs. of (B,S,P), respectively.

(1) Blue: $3x_1 + 4x_2 + 5x_3 = 225$

Then: (2) Straw: $6x_1 + 6x_2 + 6x_3 = 300 \Leftrightarrow \boxed{x_1 + x_2 + x_3 = 50}$

(3) Peach: $9x_1 + 8x_2 + 7x_3 = 375$

(3) - (1) and dividing by 2 $\Rightarrow$ (4) $3x_1 + 2x_2 + x_3 = 75$

(4) - (2) $\Rightarrow 2x_1 + x_2 = 25 \Rightarrow \boxed{x_2 = 25 - 2x_1} \Rightarrow 1 \le x_1 \le 12$

Substituting in the boxed expression, $x_1 + (25 - x_1) + x_3 = 50 \Rightarrow \boxed{x_3 = 25 + x_1} \Rightarrow 26 \le x_3 \le 37$.

On first glance, it might appear that $k = 12$, but for $k = 5,10$, x_1, x_2, and x_3 are <u>not</u> relatively prime.

Thus, $(k, M, m) = \underline{\mathbf{(10, 37, 26)}}$.

Check: $(1,23,26) \Rightarrow \begin{cases} B: 3 + 23(4) + 26(5) = 225 \\ S: 6(1 + 23 + 26) = 300 \\ P: 9 + 23(8) + 26(7) = 375 \end{cases}$, $(12,1,37) \Rightarrow \begin{cases} B: 12(3) + 4 + 37(5) = 225 \\ S: 6(12 + 1 + 37) = 300 \\ P: 12(9) + 8 + 37(7) = 375 \end{cases}$.

B) Solutions of $\sqrt{3x} - \sqrt{3x - k} > 1$ require that $3x - k \ge 0 \Rightarrow x \ge \frac{k}{3}$.

Transposing $\sqrt{3x - k}$ to the right-hand side of the inequality, $\sqrt{3x} > 1 + \sqrt{3x - k}$, we note that both sides represent nonnegative quantities, so squaring both sides will preserve the inequality.

$\Rightarrow \cancel{3x} > 1 + 2\sqrt{3x - k} + \cancel{3x} - k \Leftrightarrow \boxed{k - 1 > 2\sqrt{3x - k}}$

Since we know $k \ge 1$, we can square both sides again.

$k^2 - 2k + 1 > 4(3x - k) = 12x - 4k \Leftrightarrow 12x < k^2 + 2k + 1 \Leftrightarrow x < \frac{(k+1)^2}{12}$.

Thus, solutions are bounded. Specifically, $\frac{k}{3} \le x < \frac{(k+1)^2}{12}$.

A table is a good way to correlate various k-values with the number of integer solutions.

$k =$	1	2	3	4	5	6
Interval	$\frac{1}{3} \le x < \frac{1}{3}$	$\frac{2}{3} \le x < \frac{3}{4}$	$1 \le x < \frac{4}{3}$	$\frac{4}{3} \le x < \frac{25}{12}$	$\frac{5}{3} \le x < 3$	$2 \le x < \frac{49}{12}$
# solutions	0	0	1	1	1	3

Thus, there are never exactly **<u>2</u>** integer solutions.

Team Round - continued

C) Besides satisfying the proportion $\frac{\cos A}{b}=\frac{\cos B}{a}$, we know that the sides and angles in ΔABC satisfy the proportion $\frac{\sin A}{a}=\frac{\sin B}{b}\Rightarrow b=\frac{a\sin B}{\sin A}$. Substituting for b in the given proportion, we have $\frac{\cos A}{\frac{a\sin B}{\sin A}}=\frac{\cos B}{a}\Rightarrow \cancel{a}\sin A\cos A=\cancel{a}\sin B\cos B\Rightarrow \sin 2A=\sin 2B$.

Therefore, the arguments are either equal or supplementary (since $\sin(180°-\theta)=\sin\theta$)

$\Rightarrow\ 2A=2B$ or $2A+2B=180$

$\Rightarrow A=B$ (Δs are equilateral or isosceles) or $A+B=90$ (ABC is a right triangle)

The possibilities are - Equilateral: 8-8-8 Isosceles: 11-11-2, 10-10-4, 9-9-6, Right: 6-8-10

The equilateral triangle has the maximum area of $M=\frac{8^2\sqrt{3}}{4}=16\sqrt{3}$.

The isosceles triangle with the shortest base has minimum area.

$m=\frac{1}{2}\cdot 2\cdot\sqrt{11^2-1^2}=\sqrt{120}=2\sqrt{30}$.

Thus, $(n,M,m)=\underline{\mathbf{(5,16\sqrt{3},2\sqrt{30})}}$.

D) Rather than just converting everything to decimal, doing the computations in base 10, and converting the results back into the two other bases, let's do the computations in the given base and take advantage of the fact that all digits in base 16 can be written in terms of 4 binary digits, namely 0000 …. 1111, and that all octal digits can be written in terms of 3 binary digits, namely 000…111.

Working in base 16, $2x-7=AB5\Leftrightarrow 2x=AB5+7=ABC$

Dividing by 2, $2\overset{55E}{\overline{)AB^1C}}$ Recall: $1C_{16}=(16+12)_{10}=28$ and $\frac{28}{2}=14=E_{16}$

$55E_{16}=0101\ 0101\ 1110_2$ Regrouping in blocks of 3 binary digits,

$0101\ \ 0101\ \ 1110_2=010\ |\ 1\ 01\ |\ 01\ 1\ |\ 110_2=2536_8$

Converting $55E_{16}$ to decimal, we have $5(16^2)+5(16)+14=5(256)+80+14=1280+94=1374$.

Thus, $(o,d,h)=\underline{\mathbf{(2536,\ 1374,\ 55E)}}$.

Team Round - continued

E) The semi-perimeter of the rectangle is 119.
The diameter *d* of the circumcircle is the diagonal of the rectangle
Clearly, there are 119 integer ordered pairs (L,W) for which $L+W=119$, but the requirement that the diagonal also have integer length narrows down the list of possibilities significantly. We require that $d^2=L^2+W^2$. Since $119=7\cdot 17$, if we can find a right triangle with integer dimensions in which the sum of the lengths of the legs is 7, then scaling that right triangle by a factor of 17 will produce the correct perimeter for the rectangle.
If the sum of the legs is 17, scale by a factor of 7.
Two common Pythagorean Triples fill the bill: 3-4-5 and 5-12-13.

$17(3-4-5)=(57-68-85)$, $[51+68=119]$ and $r=\dfrac{85}{2}=\underline{\mathbf{42.5}}$

$7(5-12-13)=(35-84-91)$, $[35+84=119]$ and $r=\dfrac{91}{2}=\underline{\mathbf{45.5}}$

Are there others? Yes! Here's why. Pythagorean Triples are generated by:

$\begin{cases} a=m^2-n^2 \\ b=2mn \\ c=m^2+n^2 \end{cases}$, where *c* is the hypotenuse, and *a* and *b* are legs.

Of course, *m* must be greater than *n*. ***

We require that $a+b=m^2-n^2+2mn=\left(m^2+2mn+n^2\right)-2n^2=(m+n)^2-2n^2=119$, i.e., $119+2n^2$ must be a perfect square; so, a chart is a perfect way to organize our quest.

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
$2n^2$	2	8	18	32	50	72	98	128	162	200	242
$119+2n^2$	<u>**121**</u>	127	137	151	<u>**169**</u>	191	217	247	281	319	<u>**361**</u>
m + n	**11**				**13**						**19**
m	**10**				**8**						8, rejected
Triple	**99-20-101**				**39-80-89**						***

Thus, the remaining possibilities are $r=\dfrac{101}{2}=\underline{\mathbf{50.5}}$ and $r=\dfrac{89}{2}=\underline{\mathbf{44.5}}$.

F) The k^{th} term in the expansion of $\left(x^m+\dfrac{1}{x^2}\right)^6$ is $\dbinom{6}{k}\left(x^m\right)^{6-k}\left(x^{-2}\right)^k=\dbinom{6}{k}x^{6m-(m+2)k}$.

Thus, $6m-(m+2)k=0 \Leftrightarrow k=\dfrac{6m}{m+2}=6-\dfrac{12}{m+2}$

$m=1 \Rightarrow k=2 \Rightarrow \dbinom{6}{2}=\underline{\mathbf{15}}$ $\qquad$ $m=4 \Rightarrow k=4 \Rightarrow \dbinom{6}{4}=15$

$m=2 \Rightarrow k=3 \Rightarrow \dbinom{6}{3}=\underline{\mathbf{20}}$ $\qquad$ $m=10 \Rightarrow k=5 \Rightarrow \dbinom{6}{5}=\underline{\mathbf{6}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2018 ANSWERS

Round 1 Algebra 2: Simultaneous Equations and Determinants

A) 6 B) $-\frac{1}{3}$ C) $(4,-1,3,10)$

Round 2 Algebra 1: Exponents and Radicals

A) 0, 4 B) 30 C) -128.72

Round 3 Trigonometry: Anything

A) 166 B) $\left(\frac{15}{17},0\right)$ C) $\left(34,50,-Tan^{-1}\left(\frac{15}{8}\right)\right)$

$\theta = \pi - Tan^{-1}\left(\frac{15}{8}\right)$ is acceptable

1.875 for $\frac{15}{8}$ is acceptable.

Round 4 Algebra 1: Anything

A) 2521 B) 9.6 hours C) 2, 3

Round 5 Plane Geometry: Anything

A) 24 B) 4 C) 2000

Round 6 Algebra 2: Probability and the Binomial Theorem

A) $\frac{60}{143}$ B) $-9x$ C) 11

Team Round

A) $(10,37,26)$ D) $(2536, 1374, 55E)$

B) 2 E) 42.5, 44.5, 45.5, 50.5

C) $\left(5,16\sqrt{3},2\sqrt{30}\right)$ F) 6, 15, 20