MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2018
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) ______________________________

A) $(x, y) = (0,0)$ is a solution to the system $\begin{cases} 4kx - 3y = 0 \\ 7x + 8y = 0 \end{cases}$.
For what value of the constant k will the system have a solution other than $(x, y) = (0,0)$?

B) A linear function $y = mx + b$ is defined by the parametric equations $\begin{cases} x = \dfrac{2t-1}{3} \\ y = \dfrac{3-t}{4} \end{cases}$.

Compute the ordered pair (m, b).

C) $y = P(x)$ defines a cubic function for which $P(a) = -P(-a)$.
When $P(x)$ is divided by $(x + 3)$, the remainder is 6.
When $P(x)$ is divided by $(x - 2)$, the remainder is 4. Compute the <u>zeros</u> of $y = P(x)$.

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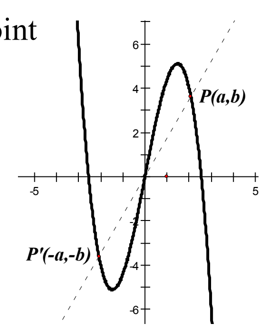
Round 1

A) This pair of lines is intersecting, parallel or coincident (the same line).
Since both lines pass through the origin, i.e. (0, 0) is a solution, only in the last case can there be another. Thus, the slopes must be equal, namely, $m=\frac{4k}{3}=\frac{-7}{8}\Rightarrow k=$ $\underline{\mathbf{-\frac{21}{32}}}$ or $\underline{\mathbf{-0.65625}}$.

Alternate solution: This system is indeterminate if and only if the determinant of the matrix of coefficients is zero. $\begin{vmatrix}4k & -3\\ 7 & 8\end{vmatrix}=0\Rightarrow 32k+21=0\Rightarrow k=$ $\underline{\mathbf{-\frac{21}{32}}}$.

B) Solving the parametric equations for *t*, we have $t=\frac{3x+1}{2}$ and $t=\frac{3-4y}{1}$.
Cross multiplying $3x+1=6-8y\Rightarrow y=mx+b=\frac{-3x+5}{8}\Rightarrow (m,b)=$ $\underline{\mathbf{\left(-\frac{3}{8},\frac{5}{8}\right)}}$.

C) Let $P(x)=Ax^3+Bx^2+Cx+D$
Since $P(a)=-P(-a)$, $B=D=0$ and $P(x)=Ax^3+Cx$.
$\begin{cases}P(-3)=-27A-3C=6\Leftrightarrow 9A+C=-2\\ P(2)=8A+2C=4\Leftrightarrow 4A+C=2\end{cases}$
Subtracting, $5A=-4\Rightarrow A=-\frac{4}{5}$. Substituting in the second equation above,
$4\left(-\frac{4}{5}\right)+C=2\Rightarrow C=\frac{26}{5}$
Thus, $P(x)=-\frac{4}{5}x^3+\frac{26}{5}x=-\frac{2}{5}x\left(2x^2-13\right)$. Setting $P(x)=0$, we have the zeros $\underline{\mathbf{0,\pm\frac{\sqrt{26}}{2}}}$.
FYI: The condition $P(-a)=-P(-a)$ defines an odd function, i.e., a function which is symmetric with respect to the origin. If the function passes through the point (a,b), then it also passes through $(-a,-b)$.
The graph of $y=P(x)$ is shown at the right.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2018
ROUND 2 ARITHMETIC / NUMBER THEORY

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) ______________________________

A) N is a two-digit prime. If the digits of N are reversed, we obtain a number with an odd number of factors. Compute the smallest possible value of N.

B) Compute the ordered pair (a, b) for which the eight-digit base ten integer $N = 437213ab$ is divisible by 72.

C) Given: x and y are positive integers, where $x < 4$ and $y < 4$.
If x is added to one of the exponents in the prime factorization of 17280, the exponents (in some order) form an *arithmetic* sequence.
If y is added to one of the exponents in the prime factorization of 17280, the exponents (in some order) form a *geometric* sequence.
A new number Q with k divisors is formed from the prime factorization of 17280 by adding x to one of the exponents and adding y to a different exponent.
Compute all possible values of k.

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Round 2

A) Since only perfect squares have an odd number of factors, we examine perfect squares until we find one which produces a prime when the digits are reversed.

$2^2 = \cancel{04}, 3^2 = \cancel{09}, 4^2 = 16 \Rightarrow \underline{\mathbf{61}}$ which is prime.

To confirm that this is the smallest value of *N*, we examine the reversals of the two-digit primes smaller than 61, namely, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59 .

None produce a perfect square. $\boxed{61} \Rightarrow 16 = 2^4 \Rightarrow 1,2,4,8,16$

B) *N* must be divisible by 8 and 9.
To guarantee divisibility by 9, the sum of all the digits must be divisible by 9.
To guarantee divisibility by 8, the 3-digit integer 3*ab* must be divisible by 8.
Thus, 20 + *a* + *b* must equal 27 or 36.
a + *b* = 7 or 16
Since *b* must be even, we consider (*a*, *b*) = (1, 6), (5, 2), (3, 4) and (8, 8).
Only **(5, 2)** satisfies divisibility by 8.

C) $17280 = 2^7 \cdot 3^3 \cdot 5^1$
Since the arithmetic sequence must be 1, 4, 7 (for *x* = 1).
Since the geometric sequence must be 1, 3, 9 (for *y* = 2).
Therefore, there are 6 possible sets of exponents of the prime factorization of *Q*:

$2,1,0 \Rightarrow \{9,4,1\} \Rightarrow 2^9 3^4 5^1 \Rightarrow 10 \cdot 5 \cdot 2 = \underline{\mathbf{100}}$ factors

$1,2,0 \Rightarrow \{8,5,1\} \Rightarrow 9 \cdot 6 \cdot 2 = \underline{\mathbf{108}}$ factors

$2,0,1 \Rightarrow \{9,3,2\} \Rightarrow 10 \cdot 4 \cdot 3 = \underline{\mathbf{120}}$ factors

$1,0,2 \Rightarrow \{8,3,3\} \Rightarrow 9 \cdot 4 \cdot 4 = \underline{\mathbf{144}}$ factors

$0,1,2 \Rightarrow \{7,4,3\} \Rightarrow 8 \cdot 5 \cdot 4 = \underline{\mathbf{160}}$ factors

$0,2,1 \Rightarrow \{7,5,2\} \Rightarrow 8 \cdot 6 \cdot 3 = 144$ factors

Thus, the 6 sets give only 5 distinct results.

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ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Compute: $\left(\sin\left(\mathrm{Cos}^{-1}(-1) - Sin^{-1}\left(-\frac{1}{2}\right)\right)\right)^3$

B) Compute: $\cot\left(Sin^{-1}\left(-\frac{3}{5}\right) - Cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$

C) Compute <u>all</u> values of *x*, where $0° \le x < 360°$, that satisfy

$$\cos(40°) - \cos(20°) = \sin(x + 30°) + \sin(x + 150°).$$

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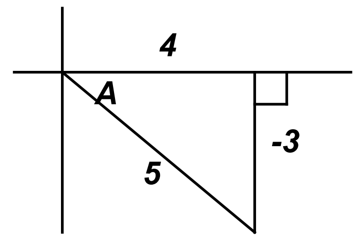
Round 3

A) $\left(\sin\left(Cos^{-1}(-1) - Sin^{-1}\left(-\tfrac{1}{2}\right)\right)\right)^3 = \left(\sin\left(\pi - \left(-\frac{\pi}{6}\right)\right)\right)^3 = \sin^3\left(\frac{7\pi}{6}\right) = \left(-\frac{1}{2}\right)^3 = \underline{\mathbf{-\frac{1}{8}}}$.

B) Let $A = Sin^{-1}\left(-\frac{3}{5}\right)$ and $B = \frac{3\pi}{4}$. Then:



$$\cot(A-B) = \frac{1+\tan A\tan\left(\frac{3\pi}{4}\right)}{\tan A - \tan\left(\frac{3\pi}{4}\right)} = \frac{1+\left(-\frac{3}{4}\right)(-1)}{\left(-\frac{3}{4}\right)-(-1)} = \frac{7/4}{1/4} = \underline{\mathbf{7}}.$$

C) $\cos(40°) - \cos(20°) = -2\sin\left(\frac{40°+20°}{2}\right)\sin\left(\frac{40°-20°}{2}\right) = -2\sin 30°\sin 10° = -\sin 10°$

$\sin(x+30°) + \sin(x+150°) = \underline{\sin x\cos 30°} + \sin 30°\cos x + \underline{\sin x\cos 150°} + \sin 150°\cos x$

Since $\cos 150° = -\cos 30°$, the underlined terms cancel out and we have

$\sin 30°\cos x + \sin 150°\cos x = 2\sin 30°\cos x = \cos x$

Therefore, the original equation simplifies to

$\cos x = -\sin 10° \Leftrightarrow \cos x = -\cos 80° \Leftrightarrow \cos x = \cos 100°$

Thus, $x = \underline{\mathbf{100}}°$ or the related value in quadrant 3, namely, $\underline{\mathbf{260}}°$, since $\cos 100°$ is negative.

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ROUND 4 ALGEBRA 1: WORD PROBLEMS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) How many cc (cubic centimeters) of a 90% solution of alcohol should be mixed with a 40% solution of alcohol to form 90 cc of a 60% solution?

B) The Red Sox win exactly 4 of their first 9 games. If they win exactly 75% of their remaining games, what is the <u>minimum</u> number of additional games they must play to raise their winning percentage to <u>at least</u> 64%?

C) I travelled from A to B at an average speed of 60 mph and returned from B to A at an average speed of 72 mph. In my haste, I forgot the keys to my apartment and had to return to B over the same route to retrieve them, travelling at an average of k mph.
My overall average for the A-B-A-B trip was 54 mph. The traffic on my last leg was just awful. Compute k.

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Round 4

A) $0.9x + 0.4(90 - x) = 0.6(90) \Rightarrow 9x + 360 - 4x = 540 \Rightarrow 5x = 180 \Rightarrow x =$ **<u>36</u>**.

B) Let x denote the number of additional games which must be played.
Note x must be a multiple of 4 to insure that the Red Sox can win exactly 75% of these games.
$4+\frac{3}{4}x \ge \frac{64}{100}(9+x) \Rightarrow 4+0.75x \ge 5.76+0.64x$
$\Rightarrow 0.11x \ge 1.76 \Rightarrow x \ge 16$
Since 16 is a multiple of 4, the minimum number of additional games is **<u>16</u>**.

C) This is not a simple $\frac{60+72+k}{3}=54$ calculation, since I travelled at these speeds for different amounts of time. A *weighted* average is required.
Pick any convenient distance for AB (any will do). I'll use the LCM(60, 72), namely 360.
I travelled $3(360)$ miles in $6+5+\frac{360}{k}$ hours $\Rightarrow \frac{1080}{11+\frac{360}{k}}=54 \Rightarrow 20=11+\frac{360}{k} \Rightarrow k=\underline{\mathbf{40}}$.

Check that $\frac{60+72+k}{3}=54 \Rightarrow k=30$ is **<u>incorrect</u>**!
@60, 72 and 30 mph, legs of 360 miles would take 6 hours, 5 hours and 12 hours respectively. The average speed would be $\frac{\text{Total distance travelled}}{\text{total time taken}}=\frac{3(360)}{6+5+12}=\frac{1080}{23}\ne 54$.
Can you show that for any arbitrary distance between A and B, $k = 30$ is incorrect?

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ROUND 5 PLANE GEOMETRY: CIRCLES

ANSWERS

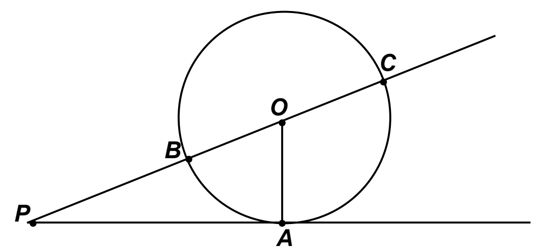
A) ______________________________

B) ______________________________

C) ______________________________

A) In circle O, $r = 30$. Chord $\overline{AE}$ is perpendicular to diameter $\overline{BD}$ at point C.
If $AE = 48$ and C is closer to D than to B, compute AD.

B) $\overrightarrow{PA}$ is tangent to circle O at point A, $PB = 1$, and $PA = BC$.
Compute OA.



C) The ring between two concentric circles has width 4.
Two perpendicular chords $\overline{AB}$ and $\overline{CD}$ are tangent to the inner circle and intersect at point P.
If $PC = 1$, compute <u>all</u> possible radii of the inner circle.

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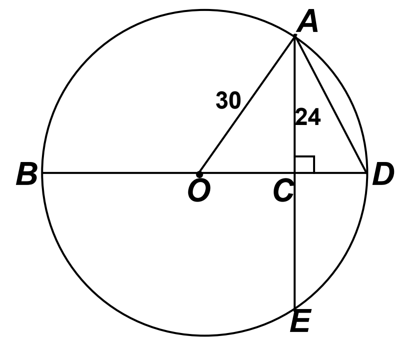
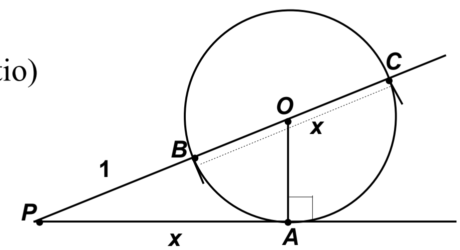
Round 5



A) In ΔAOC, $(OC, AC, AO) = (?, 24, 30) = 6(\_\_, 4, 5)$
$\Rightarrow OC = 6 \cdot 3 = 18$
$\Rightarrow CD = 12$

In ΔACD, $(AC, CD, AD) = (24, 12, ?) = 12(2, 1, \_\_)$
$\Rightarrow AD = 12 \cdot \sqrt{5} = \underline{\mathbf{12\sqrt{5}}}$.



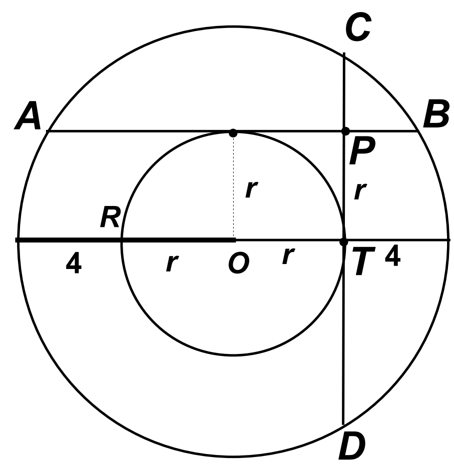
B) Using the secant-tangent relation, we have

$x^2 = 1(1+x) \Rightarrow x^2 - x - 1 = 0 \Rightarrow x = \frac{1+\sqrt{5}}{2} = g$ (the golden ratio)

$\therefore$ $\overline{OA}$ (a radius) $= \frac{1}{2} g = \underline{\mathbf{\frac{1+\sqrt{5}}{4}}}$

Alternately, by the Pythagorean Theorem,

$x^2 + \left(\frac{x}{2}\right)^2 = \left(1 + \frac{x}{2}\right)^2 \Rightarrow \frac{5}{4}x^2 = 1 + x + \frac{x^2}{4} \Rightarrow x^2 - x - 1 = 0$ and the same result follows.



C) Case 1: $PC = 1$ (P closer to C)
Applying the Product-Chord Theorem at point T,

$(r+1)^2 = 4(2r+4)$

$\Rightarrow r^2 - 6r - 15 = 0$

$\Rightarrow r = \frac{6 \pm \sqrt{36+60}}{2} = \underline{\mathbf{3+2\sqrt{6}}}$, ~~$3-2\sqrt{6}$~~

Case 2: (P closer to D) – reverse C and D in the diagram
$PC < 2r+4$ and $PC = 1 \Rightarrow r > -1.5$, but this is always true, so we proceed as follows:

Let $PD = x$ Then: $CD = x+1 \Rightarrow CT = \frac{x+1}{2}$

$\left(\frac{x+1}{2}\right)^2 = 4(2r+4) \Leftrightarrow (x+1)^2 = 32(r+2)$

$\Rightarrow PT = r = \frac{1-x}{2} \Rightarrow x = 1 - 2r$

Substituting, $(2-2r)^2 = 32(r+2) \Leftrightarrow (1-r)^2 = 8r + 16$

$\Rightarrow r^2 - 10r - 15 = 0 \Rightarrow r = \frac{10 \pm \sqrt{100+60}}{2} = 5 + 2\sqrt{10}$, ~~$5-2\sqrt{10}$~~

$r = 5 + 2\sqrt{10} \Rightarrow x < 0$ and must be rejected also.

Thus, the radius of the inner circle is unique, namely $\underline{\mathbf{3+2\sqrt{6}}}$.

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ROUND 6 ALGEBRA 2: SEQUENCES AND SERIES

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) The first term (t_1) of a sequence is 71. To find subsequent terms follow this rule:
- if a term is even, the next term is half the current term
- if a term is odd, the next term is 1 more than the current term

Eventually, t_n becomes 1. Compute the smallest value of n for which $t_n = 1$.

B) Given: $f(x) = ax^2 + bx + c$, where $f(1), f(2), f(3)$ is a geometric progression which has a constant multiplier of 3 and $f(1) + f(2) + f(3) = 13$. Compute $f(-1)$.

C) The sum of an infinite geometric series is 9 and the sum of its first two terms is 5. Compute the two possible ordered pairs (a, r), where a is the first term of the series and r is the common ratio.

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Round 6

A) Rules: if a term is even, the next term is half the current term
if a term is odd, the next term is 1 more than the current term
Applying the rules, $71 \Rightarrow 72 \Rightarrow 36 \Rightarrow 18 \Rightarrow 9 \Rightarrow 10 \Rightarrow 5 \Rightarrow 6 \Rightarrow 3 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.
Thus, the 12^{th} term has become 1 and n = **12**.

B) $f(1) = p,\ f(2) = 3p,\ f(3) = 9p \Rightarrow 13p = 13 \Rightarrow p = 1$
$f(1) = 1,\ f(2) = 3,\ f(3) = 9$
Substituting for x in $ax^2 + bx + c$,

$$\begin{cases} f(1) = a + b + c = 1 \\ f(2) = 4a + 2b + c = 3 \\ f(3) = 9a + 3b + c = 9 \end{cases} \Rightarrow \begin{cases} 3a + b = 2 \\ 5a + b = 6 \end{cases} \Rightarrow 2a = 4 \Rightarrow (a,b,c) = (2,-4,3)$$

$\Rightarrow f(-1) = a - b + c = \underline{\mathbf{9}}$

C) $$\begin{cases} a + ar + ar^2 + \ldots = 9 \\ a + ar = 5 \end{cases} \Rightarrow \begin{cases} \dfrac{a}{1-r} = 9 \\ a(1+r) = 5 \end{cases}$$

Then: $a = 9(1-r)$ Substituting,

$$9(1-r)(1+r) = 9\left(1-r^2\right) = 5 \Rightarrow 1 - r^2 = \frac{5}{9} \Rightarrow r^2 = \frac{4}{9} \Rightarrow r = \pm\frac{2}{3}$$

$$r = \frac{2}{3} \Rightarrow a = 9\left(1 - \frac{2}{3}\right) \Rightarrow (a,r) = \underline{\left(\mathbf{3}, \mathbf{\frac{2}{3}}\right)}.$$

$$r = -\frac{2}{3} \Rightarrow a = 9\left(1 + \frac{2}{3}\right) \Rightarrow (a,r) = \underline{\left(\mathbf{15}, -\mathbf{\frac{2}{3}}\right)}.$$

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____________________________

B) (______ , ______ , ______) E) ____________________________

C) ____________________________ F) ____________________________

A) The function defined by $y = f(x) = Ax^2 + Bx + C$ passes through $(5,15)$ and $(1,-17)$ and has a minimum value at $x = 2$. $y = g(x) = f(x+h)$ has a minimum at $x = -\frac{3}{2}$. Compute <u>all</u> values of x for which $g(x) = 15$.

B) Determine the unique triple of integers (a,b,c), with $a \le b \le c$, for which $\frac{17}{40} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}$.

C) Compute <u>all</u> possible values of <u>$\cos x$</u>, if $\tan\frac{x}{2} + \cot x + \csc x \sec 2x = \tan 2x$

D) The current in the Saco River is k mph, where $k < 4$.
Suppose the bionic Mr. Phelps can swim at a steady rate of 440 feet in 75 seconds.
Swimming upstream (against the current) for t minutes he can swim the same distance as he can downstream in $(t-10)$ minutes. If $t + 4k = 26$, compute this distance (in feet).

E) In ΔABC, where $\overline{AB} \perp \overline{BC}$ is a right angle, $\begin{cases} AB = x + 4y - 4 \\ AC = 7y - x + 3 \\ BC = \frac{1}{2}x + 3y - 1 \end{cases}$.

The diameter of the inscribed circle is $2x - y + 1$ and the diameter of the circumscribed circle is $3x + 2y - 4$. Compute the perimeter of ΔABC.

F) An arithmetic sequence of n terms has a first term of 2.5, a common difference of $-\frac{2}{3}$, and an integer sum S. Compute the <u>sum</u> of the <u>largest</u> five integer values of S.

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Team Round

A) $(5,15) \Rightarrow$ (#1) $25A+5B+C=15$

$(1,-17) \Rightarrow$ (#2) $A+B+C=-17$

We need a third equation. If you know calculus, you'd use a derivative; otherwise, you would complete the square. $Ax^2+Bx+C = A\left(x+\dfrac{B}{2A}\right)^2 + C - \dfrac{B^2}{4A}$

Reminiscent of the derivation of the quadratic formula!

For $y=f(x)$ to have a minimum, $A<0$.

The minimum occurs when $x+\dfrac{B}{2A}=0$ or $x=-\dfrac{B}{2A}$ and the minimum value is $C-\dfrac{B^2}{4A}$.

$-\dfrac{B}{2A}=2 \Rightarrow$ (#3) $B=-4A$

Subtracting (#1 - #2), we have $24A+4B=32 \Leftrightarrow 6A+B=8$

Substituting (using #3), we have $6A-4A=8 \Rightarrow A=4,\ B=-16,\ C=-5$

$y=4x^2-16x-5=4(x-2)^2-21$ (A minimum value of –21 at $x=2$.)

$g(x)=A(x+h)^2+B(x+h)+C=Ax^2+(2Ah+B)x+(Ah^2+Bh+C)=4x^2+8(h-2)x+(4h^2-16h-5)$

Since $y=f(x+h)$ just shifts $y=f(x)$ left or right depending on the sign of h, the minimum values of both are the same, namely –21, and the shift is from 2 to $-\dfrac{3}{2}$, that is

$h=2-\left(-\dfrac{3}{2}\right)=\dfrac{7}{2}$. Therefore, substituting for h, we have $g(x)=4x^2+12x-12$.

$g(x)=4x^2+12x-12=15 \Rightarrow 4x^2+12x-27=(2x-3)(2x+9)=0 \Rightarrow x=\underline{\boldsymbol{\dfrac{3}{2}, -\dfrac{9}{2}}}$

Note: $g(x)=4x^2+12x-12=4\left(x+\dfrac{3}{2}\right)^2-21$, confirming the minimum value is unchanged.

B) $\dfrac{17}{40} \geq \dfrac{1}{a} \Rightarrow 17a \geq 40 \Rightarrow a=3$

$\dfrac{17}{40}-\dfrac{1}{3}=\dfrac{51-40}{120}=\dfrac{11}{120}$

$\dfrac{11}{120} \geq \dfrac{1}{ab}=\dfrac{1}{3b} \Leftrightarrow 33b \geq 120 \Rightarrow b=4$

$\dfrac{11}{120}-\dfrac{1}{12}=\dfrac{1}{120}$

$\dfrac{1}{120} \geq \dfrac{1}{12c} \Rightarrow c=10 \Rightarrow (a,b,c)=\underline{\mathbf{(3,4,10)}}$

Team Round - continued

C) Given: $\tan\frac{x}{2}+\cot x+\csc x\sec 2x=\tan 2x$

$\frac{x}{2}\neq 90°+(180°)n,\ x\neq 0°+(180°)n,\ 2x\neq 90°+(180°)n$

Consolidating, excluded values (which cause division by zero) are: $0°+(180°)n$, $45°+(90°)n$

Applying basic identities,

$\tan\frac{x}{2}+\cot x+\csc x\sec 2x=\tan 2x=\frac{1-\cos x}{\sin x}+\frac{\cos x}{\sin x}+\frac{1}{\sin x\cos 2x}=\frac{\sin 2x}{\cos 2x}$

Combining terms, we have $\frac{\cos 2x+1}{\sin x\cos 2x}=\frac{\sin 2x}{\cos 2x}$.

Since $\cos 2x\neq 0$, the common term in the denominator can be ignored. Cross-multiplying, we have $\cos 2x+1=\sin 2x\sin x$

$\Leftrightarrow 2\cos^2 x=2\sin^2 x\cos x$

$\Leftrightarrow 2\cos^2 x=2(1-\cos^2 x)\cos x$

$\Leftrightarrow 2\cos x(\cos^2 x+\cos x-1)=0$

$\Leftrightarrow \cos x=\underline{\mathbf{0},\ \frac{\mathbf{-1+\sqrt{5}}}{\mathbf{2}}}$. ($\frac{-1-\sqrt{5}}{2}<-1$ and must be rejected.)

It is left to you to verify that both of these values check in the original equation.

D) $\frac{440\text{ feet}}{75\text{ sec}}=\frac{440\text{ feet}}{\frac{75}{60}\text{ min}}=440\cdot\frac{4}{5}=352\text{ ft / min}$

k mph $=\frac{k\ \cancel{\text{mi}}}{\cancel{\text{hr}}}\cdot\frac{1\ \cancel{\text{hr}}}{60\text{ min}}\cdot\frac{5280\text{ ft}}{1\ \cancel{\text{mi}}}=\frac{5280k\text{ ft}}{60\text{ min}}=88k\ \frac{\text{ft}}{\text{min}}$

Upstream: $(352-88k)t$ Downstream: $(352+88k)(t-10)$

Equating, $\cancel{352t}-88kt=\cancel{352t}-3520+88kt-880k$

$\Leftrightarrow 176kt=3520+880k\Leftrightarrow 2kt=40+10k\Rightarrow k=\frac{20}{t-5}$

Given $t+4k=26$, we substitute $k=\frac{20}{(26-4k)-5}=\frac{20}{21-4k}$

$\Rightarrow 4k^2-21k+20=(4k-5)(k-4)=0$

$k=4\Rightarrow \cancel{t=10}$ is rejected since this leaves no time to swim downstream.

Thus, $k=1.25\Rightarrow t=21$ and the distance swam upstream and downstream is

$\left(352-88\cdot\frac{5}{4}\right)(21)=242\cdot 21=$ **<u>5082</u>** feet.

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Team Round - continued

E) The perimeter P of ΔABC is $P = (x+4y-4)+(7y-x+3)+\left(\frac{1}{2}x+3y-1\right) = \frac{1}{2}x+14y-2$.

The diameter of the circumscribed circle is $\overline{AC}$, so $7y-x+3 = 3x+2y-4 \Rightarrow \boxed{4x-5y=7}$.

Since $AB+BC-AC$ is the diameter of the inscribed circle (proof below),

$(x+4y-4)+\left(\frac{1}{2}x+3y-1\right)-(7y-x+3) = 2x-y+1 \Leftrightarrow \boxed{\frac{1}{2}x+y=9}$

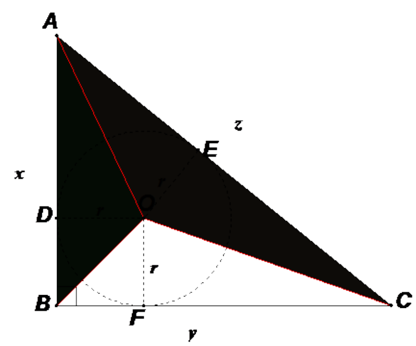
Solving the two boxed equations, $-(4x-5y=7)+8\left(\frac{1}{2}x+y=9\right) \Rightarrow 13y=65 \Rightarrow y=5, x=8$

Substituting x and y into P, we have the perimeter is $\frac{1}{2}(8)+14(5)-2 = \underline{\mathbf{72}}$.

Proof:

$|\Delta ABC| = \frac{1}{2}rx+\frac{1}{2}ry+\frac{1}{2}rz = \frac{1}{2}xy \Rightarrow r = \frac{xy}{x+y+z}$

We need to show that an equivalent expression for the diameter of the inscribed circle is $x+y-z$ or, $r = \frac{x+y-z}{2}$.



Cross-multiplying, we show that this an identity.

$2xy = (x+y-z)(x+y+z) = (x+y)^2 - z^2$

$\quad = x^2+2xy+y^2-z^2 = 2xy+\left(x^2+y^2-z^2\right)$

But, since ΔABC is a right triangle, $x^2+y^2=z^2$, and the identity is established.

F) $S = \frac{n}{2}\left(2a+(n-1)d\right) \Rightarrow S = \frac{n}{2}\left(5+(n-1)\cdot -\frac{2}{3}\right) = \frac{n}{2}\left(\frac{15-2n+2}{3}\right) = \frac{n(17-2n)}{6}$

Clearly, S can be an integer only when n is even. Testing even values of n, a pattern emerges. In successive groups of 3 consecutive even integers (starting with 2, 4, 6), the first fails and the second and third produce an integer value.

$n = 4, 6, 10, 12, 16 \Rightarrow 6, 5, -5, -14, -40 \Rightarrow S = \underline{\mathbf{-48}}$.

Since the values of S are clearly decreasing, the first 5 values listed are the largest.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2018 ANSWERS

Round 1 Algebra 2: Algebraic Functions

A) $-\frac{21}{32}$ (or -0.65625) B) $\left(-\frac{3}{8},\frac{5}{8}\right)$ C) $0,\pm\frac{\sqrt{26}}{2}$

Round 2 Arithmetic/ Number Theory

A) 61 B) (5, 2) C) 100, 108, 120, 144, 160 (in any order)

Round 3 Trig Identities and/or Inverse Functions

A) $-\frac{1}{8}$ B) 7 C) 100, 260

Round 4 Algebra 1: Word Problems

A) 36 B) 16 C) 40

Round 5 Geometry: Circles

A) $12\sqrt{5}$ B) $\frac{1+\sqrt{5}}{4}$ C) $3+2\sqrt{6}$ only

Round 6 Algebra 2: Sequences and Series

A) 12 B) 9 C) $\left(3,\frac{2}{3}\right),\left(15,-\frac{2}{3}\right)$

Team Round

A) $\frac{3}{2},-\frac{9}{2}$ D) 5082

B) $(3,4,10)$ E) 72

C) $0,\frac{-1+\sqrt{5}}{2}$ F) -48