MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2018
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

ANSWERS

A) (________ , ________ , ________)

B) Center: (_____ , _____) $r =$ ____

C) ________________________________

A) Given: $P(-6, 4)$ and $Q(2, 10)$
$\overline{PQ}$ is a diameter of a circle defined by $(x-h)^2+(y-k)^2=r^2$
Compute the ordered triple (h, k, r).

B) The ellipses $2x^2+3y^2-8x+6y-48=0$ and $3x^2+2y^2-12x+4y-52=0$ intersect in 4 points which lie on a circle. Find the center and radius of this circle.

C) Given: Hyperbola $H: \frac{(x-5)^2}{400}-\frac{(y+3)^2}{441}=1$
The horizontal line $y=3$ intersects the <u>asymptotes</u> of H at points P and Q. Compute PQ.

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Round 1

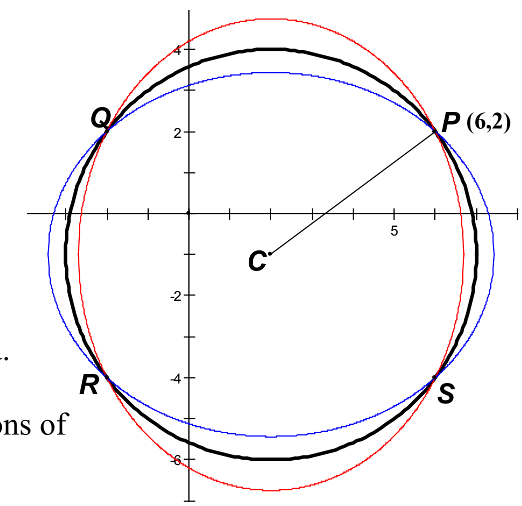
A) The center of the circle is the midpoint of $\overline{PQ}$ - $\left(\frac{-6+2}{2},\frac{4+10}{2}\right)=C(-2,7)$.

The radius of the circle is CQ (or CP).
Between points P and Q, the change in x is 8; the change in y is 6.
We recognize the Pythagorean Triple (6, 8, 10). $PQ = 10 \Rightarrow r = 5$.
Thus, (h, k, r) = **(–2, 7, 5)**.

B) Adding the equations, $\begin{cases} 2x^2+3y^2-8x+6y-48=0 \\ 3x^2+2y^2-12x+4y-52=0 \end{cases}$, we get

$5x^2+5y^2-20x+10y-100=0 \Leftrightarrow x^2+y^2-4x+2y-20=0$.
Completing the square,
$\left(x^2-4x\underline{+4}\right)+\left(y^2+2y\underline{+1}\right)=20\underline{+4+1}=25 \Leftrightarrow (x-2)^2+(y+1)^2=5^2$

$\Rightarrow$ Center: $\underline{\mathbf{(2,-1)}}$ Radius: $\underline{\mathbf{5}}$



FYI: One of the 4 points is (6, 2) and the other three can be found without too much additional effort.

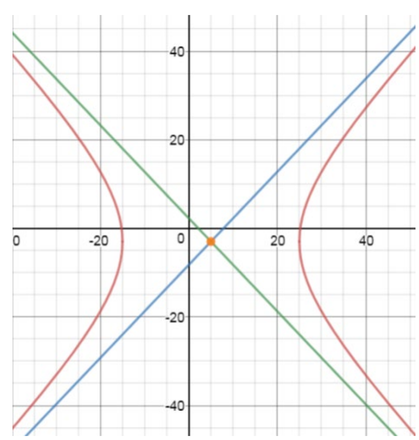
C) $\frac{(x-5)^2}{400}-\frac{(y+3)^2}{441}=1 \Leftrightarrow (y+3)^2=\frac{441}{400}(x-5)^2-441$
As $x \to \pm\infty$, the –441 is insignificant and can be ignored.

Taking the square root of both sides, we have the equations of the asymptotes!

$(y+3)=\pm\frac{21}{20}(x-5) \Rightarrow \begin{cases} 21x-20y=165 \\ 21x+20y=45 \end{cases}$

$y=3 \Rightarrow$

$x=\frac{165+60}{21},\frac{45-60}{21}=\frac{75}{7},-\frac{5}{7} \Rightarrow PQ=\frac{75}{7}-\left(-\frac{5}{7}\right)=\underline{\mathbf{\frac{80}{7}}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2018
ROUND 2 ALGEBRA 1: FACTORING AND/OR EQTNS INVOLVING FACTORING

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Determine all numbers for which one more than the square of the number is the same as 7 more than 5 times the number.

B) The integer *N* has more than 8 positive factors, but $N+1$ has only two.
If $N > 20$, compute the minimum value of *N*.

C) For some integer value of *a*, $(x-a)$ is a factor of $8x^3-40x^2+66x-36$.
Determine the complete factorization of $8x^3-40x^2+66x-36$.

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Round 2

A) $x^2+1=5x+7 \Leftrightarrow x^2-5x-6=(x+1)(x-6)=0 \Rightarrow x=\underline{\mathbf{-1,\ 6}}$.

B) We must find a pair of consecutive integers where the first is composite with the requisite number of factors and the second is prime.
Consider only numbers immediately preceding a prime, e.g., 22, 28, 30, etc.
23: $22=2\cdot 11$, only 4 factors - rejected
29: $28=2^2\cdot 7 \Rightarrow 1,2,4,7,14,28$
31: $30=2\cdot 3\cdot 5 \Rightarrow 1,2,3,5,6,10,15,30$ - rejected
37: $36=2^2\cdot 3^2 \Rightarrow 1,2,3,4,6,9,12,18,36$ - BINGO! $N=\underline{\mathbf{36}}$.
FYI: To find the number of factors of any integer (assuming the prime factorization of the number is known or easily determined), add 1 to each exponent and take the product of all of these sums, namely, $(e_1+1)\cdot(e_2+1)\cdot(e_3+1)\cdot\ldots\cdot(e_k+1)$.
Thus, without enumerating, $360=2^3\cdot 3^2\cdot 5^1 \Rightarrow (3+1)(2+1)(1+1)=4\cdot 3\cdot 2=24$ factors.

C) Factoring out the constant factor, we have
$8x^3-40x^2+66x-36=2\left(4x^3-20x^2+33x-18\right)=2(x-a)(4x^2+\ldots)$
Using synthetic substitution confirms that for $a=2$ the remainder is zero, and we have the linear and quadratic factors.

2	4	−20	33	−18
		8	−24	18
	4	−12	9	0

$2(x-2)\left(4x^2-12x+9\right)=\underline{\mathbf{2(x-2)(2x-3)^2}}$.

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ROUND 3 TRIG: EQUATIONS WITH A REASONABLE NUMBER OF SOLUTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) If $2-3\sin x=4$ and $180° < x < 270°$, compute $\cos x$.

B) Compute the secondsmallestpositive value of x (in radians) for which

$$\sin(x)=\cos\left(2x-\frac{\pi}{6}\right)$$

C) Solve over $[0°, 360°)$. $\dfrac{\sin 2x}{1-\cos x}=\tan x$

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Round 3

A) $2-3\sin x = 4 \Rightarrow \sin x = -\dfrac{2}{3}$

Since $\sin^2 x + \cos^2 x = 1$, $\cos^2 x = 1 - \dfrac{4}{9} = \dfrac{5}{9} \Rightarrow \cos x = \pm\dfrac{\sqrt{5}}{3}$

But $180° < x < 270°$ puts x in quadrant 3, where $\cos x < 0$. Thus, $\cos x = \underline{\boldsymbol{-\dfrac{\sqrt{5}}{3}}}$ only.

B) Converting the cosine to sine,

$$\sin(x) = \cos\left(2x - \frac{\pi}{6}\right) \Leftrightarrow \sin(x) = \sin\left(\frac{\pi}{2} - \left(2x - \frac{\pi}{6}\right)\right) = \sin\left(\frac{2\pi}{3} - 2x\right)$$

Since the period of the sine function is 2π, we have

$$x = \frac{2\pi}{3} - 2x + 2n\pi \Leftrightarrow 3x = \frac{2\pi}{3} + 2n\pi \Leftrightarrow x = \frac{2\pi}{9} + \frac{2n\pi}{3} = \frac{2\pi(3n+1)}{9}$$

$n = 0$ produces the smallest positive solution. $n = 1$ produces the second smallest solution $\underline{\boldsymbol{\dfrac{8\pi}{9}}}$.

Alternate solution:
The sine and cosine are cofunctions means that the values of the functions are equal if the arguments add to $\dfrac{\pi}{2}$ (i.e. are complementary)
By adding 2π to the sum we have the second largest positive answer.

$x + \left(2x - \dfrac{\pi}{6}\right) = \dfrac{\pi}{2} + 2\pi = \dfrac{5\pi}{2} \Rightarrow 3x = \dfrac{5\pi}{2} + \dfrac{\pi}{6} = \dfrac{8\pi}{3}$ and we have the same result.

C) $\dfrac{\sin 2x}{1-\cos x} = \dfrac{2\cancel{\sin x}\cos x}{1-\cos x} = \dfrac{\cancel{\sin x}}{\cos x}$

Provided $\cos x \neq 0, 1$ $(x \neq 0°, 90°, 270°)$, we can cross multiply

$$2(\cos x)^2 = 1 - \cos x \Leftrightarrow 2\cos^2 x + \cos x - 1 = (2\cos x - 1)(\cos x + 1) = 0$$

$\Rightarrow \cos x = \dfrac{1}{2}, -1 \Rightarrow x = \underline{\mathbf{60°, 300°, 180°}}$ (All values will check!)

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 4 ALGEBRA 2: QUADRATIC EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) A curve is represented by an equation that indicates that x is inversely proportional to y. If $(-3, 4)$ is a point on this curve, compute the coordinates of <u>all</u> points of intersection of this curve and the line $y = -x$.

B) Given: B is a positive integer
The quadratic equation $25x^2 - Bx + 1 = 0$ has two <u>distinct</u> rational roots.
What is the only possible value of B?

C) Solve for x: $2 - \sqrt{5x+9} = \sqrt{8x+17}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2018 SOLUTION KEY

Round 4

A) x inversely proportional to $y \Rightarrow x = k\left(\frac{1}{y}\right) \Rightarrow xy = k$

$(-3, 4)$ is a point on the curve $\Rightarrow k = -12$

$\begin{cases} xy = -12 \\ y = -x \end{cases} \Rightarrow x^2 = 12 \Rightarrow x = \pm 2\sqrt{3} \Rightarrow \underline{\mathbf{\left(-2\sqrt{3}, 2\sqrt{3}\right), \left(2\sqrt{3}, -2\sqrt{3}\right)}}$

B) The only possible factorizations are $(5x-1)^2$ and $(25x-1)(x-1)$. Since the first does not yield two distinct roots, the second is the required factorization and $B = \underline{\mathbf{26}}$.

Alternate (but more difficult) solution:

The roots are $\frac{B \pm \sqrt{B^2 - 100}}{2 \cdot 25}$. Two distinct rational roots requirethat the discriminant be a positive perfect square. That leaves out $B = 10$ which produces a discriminant of zero and exactly one root, $x = 5$.

We can experiment by trial and error or…… analyze the situation as follows:

Suppose $B^2 - 100 = k^2$, for some positive integer k.

Transposing terms, we have the difference of perfect squares: $B^2 - k^2 = 100$.

Factoring, $(B+k)(B-k) = 100 \cdot 1,\ 50 \cdot 2,\ 25 \cdot 4,\ 20 \cdot 5$ or ~~$10 \cdot 10$~~

Equating factors and adding, we have $2B = \ldots$. and the term on the right side must be even.

The only possibility is $\begin{cases} B + k = 50 \\ B - k = 2 \end{cases} \Rightarrow B = \underline{\mathbf{26}}$.

C) Squaring both sides, $4 - 4\sqrt{5x+9} + 5x + 9 = 8x + 17 \Rightarrow -4\sqrt{5x+9} = 3x + 4$

$\Rightarrow 16(5x+9) = 9x^2 + 24x + 16 \Rightarrow 9x^2 - 56x - 128 = (9x+16)(x-8) = 0$

$\Rightarrow x = \underline{\mathbf{-\frac{16}{9}}}$

8 is extraneous, since $2 - \sqrt{49} \neq \sqrt{81}$

Here is the check for $x = -\frac{16}{9}$:

$$\left(2 - \sqrt{\frac{-80}{9} + \frac{81}{9}} = 2 - \sqrt{\frac{1}{9}} = \frac{5}{3},\ \sqrt{\frac{-128}{9} + \frac{153}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}\right)$$

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

ANSWERS

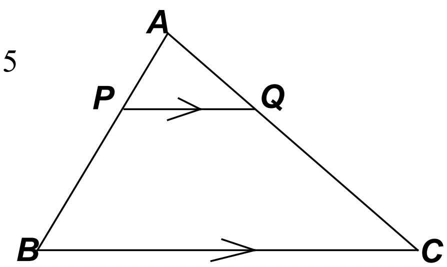
A) ________________________________

B) ________________________________

C) ________________________________

A) Given: $\overline{PQ} \parallel \overline{BC}$, area($\Delta PAQ$) = 8.5 and area($PQCB$) = 42.5

Compute $\dfrac{AP}{AB}$.



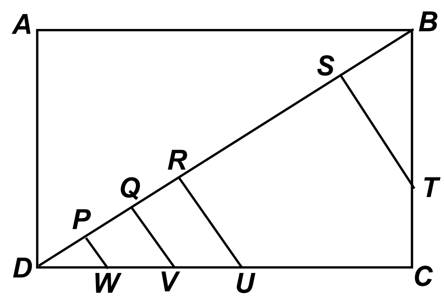
B) Given:

$ABCD$ is a rectangle

$BC = AB - 2 = BD - 4$, $PW = 1$, $QV = TC = 2$, $RU = 3$

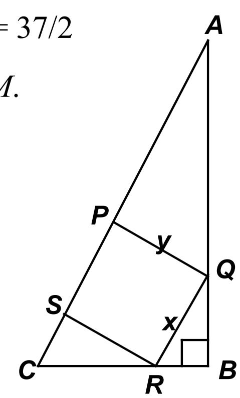
$\overline{PW}$, $\overline{QV}$, $\overline{RU}$ and $\overline{ST}$ are all perpendicular to $\overline{BD}$.

Compute RS.



C) Given: $PQRS$ is a rectangle, $BC : AB : AC = 12 : 35 : 37$, Per($\Delta ABC$) = 420, $x = 37/2$

In simplified form, $y = \dfrac{M}{N}$. Compute the <u>sum</u> of the distinct prime factors of M.



MASSACHUSETTS MATHEMATICS LEAGUE
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Round 5

A) $42.5 = 5(8.5) \Rightarrow$ area(ΔPAQ) : area(ΔBAC) = 1 : 6 $\Rightarrow \dfrac{AP}{AB} = \dfrac{1}{\sqrt{6}} = \underline{\mathbf{\dfrac{\sqrt{6}}{6}}}$.

B) If $AB = x$, then $BC = x-2,\ BD = x+2$.

$x^2 + (x-2)^2 = (x+2)^2 \Rightarrow x^2 - 8x = 0 \Rightarrow x = 8 \Rightarrow (BC, AB, BD) = (6,8,10)$.

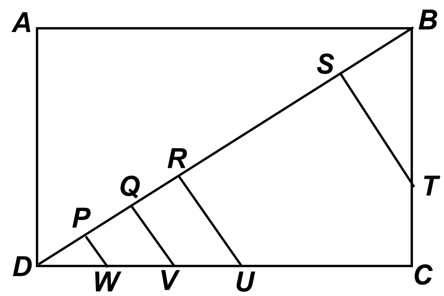
Since $\Delta DPW \sim \Delta DQV \sim \Delta DRU \sim \Delta DCB$, the corresponding sides are in a ratio of 3 : 4 : 5.

$\Rightarrow DP = PQ = QR = \dfrac{4}{3}$.

Since $\Delta DCB \sim \Delta TSB$ and $TC = 2$, $BT = 4$.

$\Rightarrow SB = \dfrac{3}{5} \cdot 4 = \dfrac{12}{5} = 2.4$.

$\therefore RS = 10 - 3\left(\dfrac{4}{3}\right) - 2.4 = \underline{\mathbf{3.6}}$.

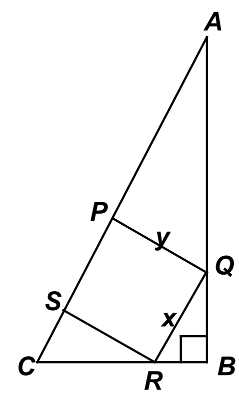


C) $(12 + 35 + 37)n = 420 \Rightarrow 84n = 420 \Rightarrow n = 5 \Rightarrow BC = 60$.

$\Delta RBQ \sim \Delta CSR \sim \Delta CBA \Rightarrow \dfrac{RQ}{CR} = \dfrac{BQ}{SR} \Rightarrow \dfrac{x}{60 - \frac{12}{37}x} = \dfrac{\frac{35}{37}x}{y}$.

Switching the means, $\dfrac{x}{\frac{35}{37}x} = \dfrac{60 - \frac{12}{37}x}{y}$.

Canceling and substituting $x = \dfrac{37}{2}, \dfrac{37}{35} = \dfrac{54}{y} \Rightarrow y = \underline{\dfrac{35(54)}{37} = \dfrac{2 \cdot 3^3 \cdot 5 \cdot 7}{37}} \Rightarrow 2 + 3 + 5 + 7 = \underline{\mathbf{17}}$.



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ROUND 6 ALGEBRA 1: ANYTHING

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) The Newton family saved D dollars for vacation expenses. Their plan was to save $\frac{1}{3}$ for food, $\frac{1}{5}$ for transportation costs, $\frac{2}{5}$ for hotels and the remaining \$400 for unforeseen expenses. Compute D.

B) Let $R(n)$ denote the value of n rounded to the nearest integer.
Compute $R\left(\sqrt{179}\right)+R\left(\sqrt{311}\right)$.

C) If $x \neq y$ and $\frac{x}{y}+x=\frac{y}{x}+y$, compute the numerical value of $\frac{1}{x}+\frac{1}{y}$.

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Round 6

A) Let D denote the total amount saved. Then:
$\frac{1}{3}D+\frac{1}{5}D+\frac{2}{5}D+400=D \Rightarrow 5D+3D+6D+6000=15D$
$\Rightarrow D=\underline{\mathbf{6000}}$.

B) Trap each radicand between consecutive perfect squares.
$13^2=169<179<196=14^2$
Since 179 is closer to 169 than 196, $\sqrt{179}$ is closer to 13, i.e. $\sqrt{179}<13.5 \Rightarrow R\left(\sqrt{179}\right)=13$
$19^2=289<311<324=18^2$
Since 311 is closer to 324 than 289, $\sqrt{311}$ is closer to 18, i.e. $\sqrt{311}>17.5 \Rightarrow R\left(\sqrt{311}\right)=18$
Thus, $R\left(\sqrt{179}\right)+R\left(\sqrt{311}\right)=\underline{\mathbf{31}}$.

C) $\frac{x}{y}+x=\frac{y}{x}+y \Leftrightarrow \frac{x}{y}-\frac{y}{x}+x-y=0 \Leftrightarrow \frac{x^2-y^2}{xy}+x-y=0 \Leftrightarrow \frac{x^2-y^2+(x-y)xy}{xy}=0$
$\Leftrightarrow \frac{(x-y)\left(x+y+xy\right)}{xy}=0$
Cross multiplying, $(x-y)\left(x+y+xy\right)=0$.
Since $x \neq y$, $\left(x+y+xy\right)=0 \Rightarrow x+y=-xy$.
Dividing by xy, $\frac{x+y}{xy}=\frac{1}{y}+\frac{1}{x}=\underline{\mathbf{-1}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________ D) (______ , ______ , ______)

B) ____________________ E) __________ : __________

C) ____________________ F) ____________________

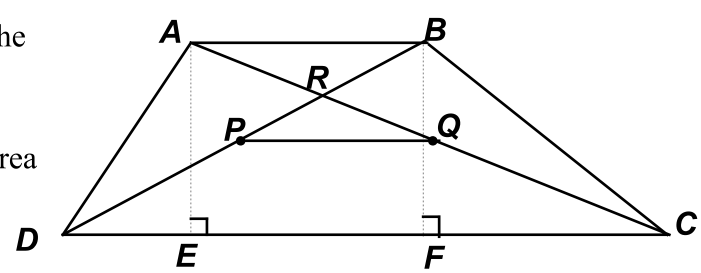
A) The line $3x+4y=k$ is tangent to $x^2+2y^2=3$. Compute <u>all</u> possible values of *k*.

B) Given: $(x,y)=\left(\frac{\sqrt{6}+\sqrt{2}}{4},\frac{\sqrt{6}-\sqrt{2}}{4}\right)$ Compute $\left(\frac{x}{x-y}\right)^3-\left(\frac{y}{x-y}\right)^3$.

C) Solve for x over $0° \le x < 360°$. $\cos^6 x-\sin^6 x=0$

D) Given: $Ax^2+Bx+C=0$, where *A*, *B*, and *C* are integers, $A>0$, $A=3B-1=2-3C$.
The sum of the coefficients equals 200 times the sum of the roots times the product of the roots.
Compute the ordered triple (A,B,C).

E) In trapezoid *ABCD*, *P* and *Q* are midpoints of the diagonals, $PQ=10.5$, $\frac{FC}{DE}=\frac{5}{2},\frac{AB}{FC}=\frac{4}{5}$.
Compute the ratio of the area of ΔPQR to the area of *ABCD*.



F) <u>How many</u> integer solutions are there to the equation $|x-7|+|x+5|\le 16$?

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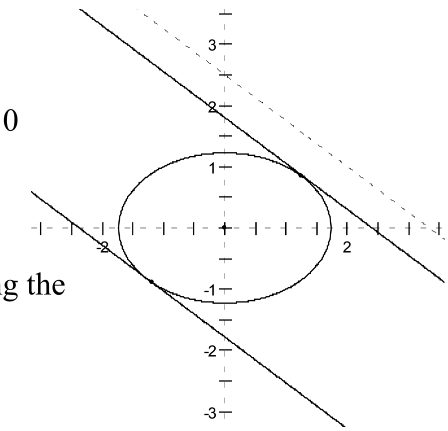
Team Round

A) $3x+4y=k \Rightarrow y=\dfrac{k-3x}{4}$



Substituting, $x^2+2\left(\dfrac{k-3x}{4}\right)^2=3 \Leftrightarrow 8x^2+k^2-6kx+9x^2-24=0$

$\Leftrightarrow 17x^2-6kx+\left(k^2-24\right)=0$. For any intersection point between the line and the curve, the x-coordinate is unique! Therefore, this quadratic equation will have equal roots, implying the discriminant must be 0.

$(-6k)^2-4(17)\left(k^2-24\right)=0 \Rightarrow -8k^2=-17\cdot 24 \Rightarrow k=\underline{\pm\sqrt{\mathbf{51}}}$.

B) $\left(\dfrac{x}{x-y}\right)^3-\left(\dfrac{y}{x-y}\right)^3=\dfrac{x^3-y^3}{(x-y)^3}=\dfrac{(x-y)\left(x^2+xy+y^2\right)}{(x-y)^3}=\dfrac{x^2+xy+y^2}{(x-y)^2}=\dfrac{\left(x^2-2xy+y^2\right)+3xy}{(x-y)^2}$

$=\dfrac{(x-y)^2+3xy}{(x-y)^2}=1+\dfrac{3xy}{(x-y)^2}$

Since $3xy=3\left(\dfrac{6-2}{16}\right)=\dfrac{3}{4}$ and $(x-y)^2=\left(\dfrac{\sqrt{2}}{2}\right)^2=\dfrac{1}{2}$, we have $1+\dfrac{3}{2}=\underline{\dfrac{\mathbf{5}}{\mathbf{2}}}$.

C) $\cos^6 x-\sin^6 x=0 \Leftrightarrow \sin^6 x=\cos^6 x$

Provided $\cos x\neq 0$, we can divide both sides by $\cos^6 x$, getting $\tan^6 x=1$.

$\Rightarrow \tan x=\pm 1$

$\Rightarrow x=$ **<u>45</u>°, <u>135</u>°, <u>225</u>°, <u>315</u>°**

None of these answers will be extraneous, since none will cause the cos x to equal zero.

Alternate solution:

$\cos^6 x-\sin^6 x=\left(\cos^3 x-\sin^3 x\right)\left(\cos^3 x+\sin^3 x\right)$

$=\left(\cos x-\sin x\right)\left(\cos^2 x+\cos x\sin x+\sin^2 x\right)\left(\cos x+\sin x\right)\left(\cos^2 x-\cos x\sin x+\sin^2 x\right)$

$=\left(\cos x-\sin x\right)\left(1+\cos x\sin x\right)\left(\cos x+\sin x\right)\left(1-\cos x\sin x\right)$

$=\left(\cos^2 x-\sin^2 x\right)\left(1-\cos^2 x\sin^2 x\right)=\left(\cos 2x\right)\left(1-\cos^2 x\sin^2 x\right)=0$

$\cos 2x=0 \Rightarrow 2x=90+180n \Rightarrow x=45+90n \Rightarrow$ **<u>45°, 135°, 225°, 315°</u>**

$\left(1-\cos^2 x\sin^2 x\right)=0 \Rightarrow \cos x\sin x=\pm 1 \Rightarrow \dfrac{\sin 2x}{2}=\pm 1 \Rightarrow \sin 2x=\pm 2$ (which has no solutions)

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Team Round - continued

D) Given: $Ax^2 + Bx + C = 0$, where *A*, *B*, and *C* are integers, $A > 0$, $A = 3B - 1 = 2 - 3C$

$$A + B + C = A + \frac{A+1}{3} + \frac{2-A}{3} = \frac{3A+3}{3} = A + 1$$

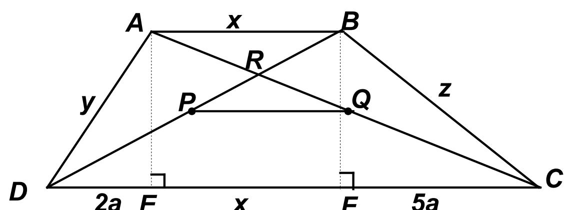
So $A + 1 = 200 \cdot \left(-\frac{B}{A}\right)\left(\frac{C}{A}\right) = 200\left(-\frac{A+1}{3A}\right)\left(\frac{2-A}{3A}\right)$

Dividing through by $A+1$, $1 = 200\left(\frac{1}{3A}\right)\left(\frac{A-2}{3A}\right) \Rightarrow 9A^2 = 200A - 400$

$\Leftrightarrow 9A^2 - 200A + 400 = 0$ Challenging to factor? Not Really!

Since we were given that *A* was an integer, one factor must be of the form $(A - ?)$.

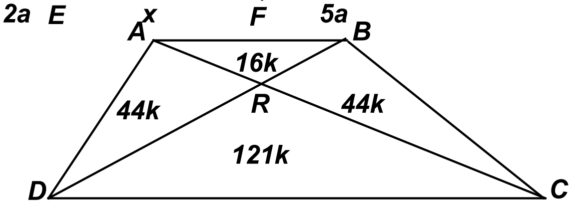
$\Leftrightarrow (A-20)(9A-20) = 0$. Therefore, $(A,B,C) = \underline{\mathbf{(20, 7, -6)}}$.



E) Given: $PQ = 10.5$, $\frac{FC}{DE} = \frac{5}{2}$, $\frac{AB}{FC} = \frac{4}{5}$

$$PQ = \frac{CD - AB}{2} = \frac{x + 7a - x}{2} = 10.5 \Rightarrow a = 3$$

$$\frac{AB}{FC} = \frac{4}{5} \Rightarrow AB = 12$$



$$\Delta BAR \sim \Delta DCR \Rightarrow \frac{AB}{CD} = \frac{12}{33} = \frac{4}{11} \Rightarrow \frac{\text{area}(\Delta BAR)}{\text{area}(\Delta DCR)} = \frac{16}{121}$$

ΔDAR and ΔBAR are NOT similar, but they have the same altitude from point *A*. Additionally, bases $\overline{DR}$ and $\overline{BR}$ are in a 11 : 4 ratio, so the areas of these triangles are also in a 11 : 4 ratio.

A similar argument establishes that the areas of ΔCBR and ΔBAR are also in a 11 : 4 ratio (which implies ΔDAR and ΔCBR although not congruent have the same area).

Note that the diagram at the right maintains these area ratios.

$$\Delta BAR \sim \Delta PQR \Rightarrow \frac{PQ}{AB} = \frac{12}{10.5} = \frac{8}{7} \Rightarrow \frac{\text{area}(\Delta BAR)}{\text{area}(\Delta PQR)} = \frac{64}{49} \Rightarrow area(\Delta PQR) = \frac{49}{4}k$$

Thus, the required ratio is $\frac{49/4}{16 + 121 + 2(44)} = \frac{49}{4(225)} = \underline{\mathbf{\frac{49}{900}}}$

FYI: We could have used the perimeter as follows:

$\text{Per} = 12 + (6 + 12 + 15) + y + z = 72 \Rightarrow y + z = 27$. Let $h = AE = BF$.

Appealing to right triangles *ADE* and *BFC*, $h^2 = y^2 - 6^2 = z^2 - 15^2 \Rightarrow z^2 - y^2 = 189$.

Substituting for *z*, $(27 - y)^2 + y^2 = 189 \Rightarrow 729 - 54y = 189 \Rightarrow y = 10,\ z = 17,\ h = 8$.

Thus, $225k = \frac{1}{2} \cdot 8 \cdot (12 + 33) = 180 \Rightarrow k = \frac{4}{5} \Rightarrow \frac{\frac{49}{4} \cdot \frac{4}{5}}{180} = \underline{\mathbf{\frac{49}{900}}}$.

Team Round - continued

F) Consider equivalent equations without absolute value.
Case 1: $x \le -5$
$7 - x + (-x - 5) \le 16 \Leftrightarrow 2x \ge -14 \Leftrightarrow x \ge -7 \Rightarrow (-7, \ldots -5)$: 3 solutions
Case 2: $-5 < x < 7$
$7 - x + (x + 5) \le 16 \Leftrightarrow 12 \le 16$
Since this inequality is always true, all integers in the interval are solutions $(-4 \ldots 6)$: 11 solutions
Case 3: $x \ge 7$
$x - 7 + (x + 5) \le 16 \Leftrightarrow 2x \le 18 \Leftrightarrow x \le 9 \Rightarrow (7 \ldots 9)$: 3 solutions
Total number of solutions: **17**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY2018 ANSWERS

Round 1 Analytic Geometry: Anything

A) $(-2, 7, 5)$ B)Center: $(2, -1)$ Radius: 5 C) $\frac{80}{7}$

Round 2 Algebra: Factoring

A) $-1, 6$ B)36 C) $2(x-2)(2x-3)^2$

Round 3 Trigonometry: Equations

A) $-\frac{\sqrt{5}}{3}$ B) $\frac{8\pi}{9}$ C) $60°, 180°, 300°$

Round 4 Algebra 2: Quadratic Equations

A) $\left(-2\sqrt{3}, 2\sqrt{3}\right), \left(2\sqrt{3}, -2\sqrt{3}\right)$ B) 26 C) $-\frac{16}{9}$

Round 5 Geometry: Similarity

A) $\frac{\sqrt{6}}{6}$ B)3.6 (or $\frac{18}{5}$) C) 17

Round 6 Algebra 1: Anything

A) 6000 B)31 C) -1

Team Round

A) $\pm\sqrt{51}$

B) $\frac{5}{2}$ or 2.5

C) $45°, 135°, 225°, 315°$

D) $(20, 7, -6)$

E) 49 : 900

F) 17