MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

ANSWERS

A) ______________________________

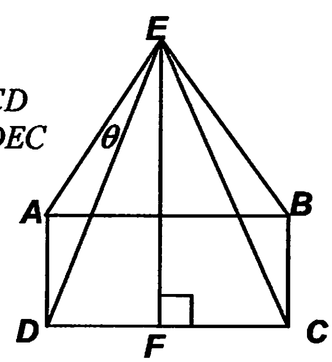
B) ______________________________

C) ______________________________

A) In ΔABC, $a^2 = b^2 + c^2 + bc\sqrt{3}$. Compute $\csc 3A$.

B) The equations $\begin{cases} m^2 - n^2 = a \\ 2mn = b \\ m^2 + n^2 = c \end{cases}$ will generate <u>primitive</u> Pythagorean Triples, for integers m and n, whenever $m > n$ and m and n are relatively prime. For example, $(m,n) = (2,1) \Rightarrow (a,b,c) = (3,4,5)$ and $(m,n) = (4,3) \Rightarrow (a,b,c) = (7,24,25)$.
Compute the perimeters of the <u>two</u> possible right triangles whose sides form a primitive Pythagorean Triple and whose hypotenuse has length 145.
Note: Two integers are relatively prime if (and only if) their greatest common factor is 1.

C) Given: In the coplanar diagram at the right, rectangle $ABCD$ has $AB = 12$ and $AD = 4$. Isosceles triangle DEC has base $\overline{DC}$ and perimeter 32.
Compute $\sin\theta$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017 SOLUTION KEY

Round 1

A) By the Law of Cosines, $a^2 = b^2 + c^2 - 2bc\cos A$. Therefore,

$-2bc\cos A = bc\sqrt{3} \Rightarrow \cos A = -\frac{\sqrt{3}}{2} \Rightarrow A = 150°$.

$\csc 3A = \frac{1}{\sin 450°} = \frac{1}{\sin(90° + 360°)} = \frac{1}{\sin 90°} = \underline{\mathbf{1}}$.

B) Since $144 + 1 = 64 + 81 = 145$, $(m,n) = (12,1), (9,8)$.

Using the Pythagorean Triple generating equations $\begin{cases} m^2 - n^2 = a \\ 2mn = b \\ m^2 + n^2 = c \end{cases}$, we have

$(12,1) \Rightarrow 143, 24, 145 \Rightarrow \text{Per} = \underline{\mathbf{312}}$.

$(9,8) \Rightarrow 17, 144, 145 \Rightarrow \text{Per} = \underline{\mathbf{306}}$.

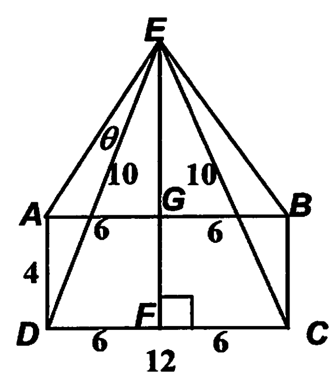
C) Since ΔDEC was given as isosceles, $DE = EC = \frac{32 - 12}{2} = 10$.

$DF = 6 \Rightarrow EF = 8$, $AD = 4 \Rightarrow EG = 4 \Rightarrow AE = \sqrt{4^2 + 6^2} = 2\sqrt{13}$

In ΔAED, $\frac{\sin\theta}{4} = \frac{\sin ADE}{2\sqrt{13}}$.

But, as complementary angles, $\cos EDF = \frac{3}{5} \Rightarrow \sin ADE = \frac{3}{5}$.

Therefore, using the law of sines in ΔAED, substituting and cross multiplying, we have $\sin\theta = \frac{4\left(\frac{3}{5}\right)}{2\sqrt{13}} = \frac{6}{5\sqrt{13}} = \underline{\frac{6\sqrt{13}}{65}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017
ROUND 2 ARITHMETIC/NUMBER THEORY

ANSWERS

A) ______________________

B) ______________________

C) ______________________

A) Exactly one of the following integers is prime. Compute the sum of the digits of this prime.
10139 2583 2883 5583 6349

B) The difference between the cubes of two consecutive positive integers is 469.
Compute the smaller of the two integers.

C) Compute the greatest common divisor of 966 and 1078.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017 SOLUTION KEY

Round 2

A) For 2583, the sum of the digits is 18, implying 2583 is divisible by 3.
Since 2883 = 2583 + 300, 2883 is also divisible by 3.
Since 5583 = 2583 + 3000, 5583 is also divisible by 3.
6349 = 7(907)
Thus, 10139 must be the prime and the required digit sum is **<u>14</u>**.

B) Let x denote the smaller number. Then:

$$(x+1)^3 - x^3 = 469 \Rightarrow \cancel{x^3} + 3x^2 + 3x + 1 - \cancel{x^3} = 469$$
$$\Rightarrow 3x^2 + 3x - 468 = 0$$
$$\Rightarrow x^2 + x - 156 = 0$$
$$\Rightarrow (x+13)(x-12) = 0$$
$$\Rightarrow x = \underline{\mathbf{12}}$$

If cubing a binomial is not yet one of your reflex reactions, you could have proceeded as follows:

$$(x+1)^2 = (x+1)^2(x+1) = (x^2+2x+1)(x+1) = (x^3+2x^2+x)+(x^2+2x+1)$$

Combining like terms, we have the above result x^3+3x^2+3x+1.

C) Brute Force:

- Find the prime factorization of each number.
- Take the product of the <u>common</u> primes, each raised to the <u>*smallest*</u> exponent to which that prime occurs in the prime factorization of either number

Note: 966 is divisible by both 2 and 3.
1078 is divisible by 2 and 11
A number is divisible by 11 if and only if the difference between the sum of the digits in even positions and the sum of the digits in odd positions is a multiple of 11.
$[(8 + 0) - (7 + 1) = 0 = 0 \cdot 11]$

$966 = 6(161) = 6 \cdot 7 \cdot 23 \Rightarrow 2^1 3^1 7^1 23^1$

$1078 = 2(539) = 2 \cdot 11 \cdot 49 = 2^1 7^2 11^1$

Therefore, the greatest common divisor is $2^1 7^1 = \underline{\mathbf{14}}$.

For larger numbers, finding the prime factorization of each number can be extremely difficult and time consuming. There is a much easier technique - it's called the Euclidean Algorithm.
It basically divides the larger number (call it *N*) by the smaller number (call it *D*) and records a quotient (*Q*) and a remainder (*R*). Then:
D becomes the new *N*-value.
Q becomes the new *D*-value.
The process is repeated until the remainder becomes 0.
The <u>*last nonzero remainder*</u> is the greatest common divisor.
Ask your coach, if you are unfamiliar with the E.A.
Here's the details of the E.A. for 966 and 1078:

N	$= DQ$	$+ R$
1078	$= 966 \cdot 1$	+ 112
966	$= 112 \cdot 8$	+ 70
112	$= 70 \cdot 1$	+ 42
70	$= 42 \cdot 1$	+ 28
42	$= 28 \cdot 1$	+ **<u>14</u>**
28	$= 14 \cdot 2$	+ 0

Practice: Compare finding the GCF of 6622 and 71595 using prime factorization and using the E.A.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

ANSWERS

A) _______________________

B) _______________________

C) _______________________

A) Given: $A(0, 52)$, $B(39, 0)$
How many lattice points are there on $\overline{AB}$, <u>excluding</u> the endpoints?

B) Two circles are internally tangent at $A(5,-2)$.
Point $B(-7,14)$, diametrically opposite point A on the smaller circle, is also the center of the larger circle. Compute the area of the region between the two circles.

C) Tangent segments are drawn from $P(1,-8)$ to the circle C whose equation is $(x-6)^2+(y-4)^2=9$. Let Q and R be the points of intersection.
Determine the equation of $\overleftrightarrow{QR}$ in standard form, i.e. $Ax+By+C=0$, where A, B and C are integers and $A>0$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017 SOLUTION KEY

Round 3

A) The slope of $\overline{AB}$ is $\frac{52-0}{0-39}=-\frac{\cancel{13}\cdot 4}{\cancel{13}\cdot 3}=\frac{-4}{3}$

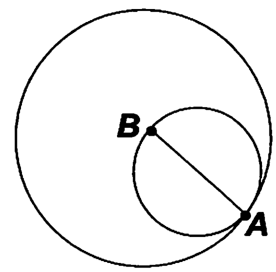
Thus, starting at A, decreasing y by 4 and increasing x by 3 produces all lattice points.
Specifically, the lattice points are of the form $(0+3k,\ 52-4k)$, where $1\le k<13$.
This produces <u>**12**</u> lattice points.

B) Given: $A(5,-2), B(-7,14)$ Using the point-to-point distance formula,

$AB^2=(5-(-7))^2+(-2-14)^2=12^2+16^2=144+256=400 \Rightarrow AB=20$

$\Rightarrow r_{\text{small}}=10,\ r_{\text{big}}=20$

Therefore, the area of the required region is $\pi(20)^2-\pi(10)^2=\underline{\mathbf{300\pi}}$.



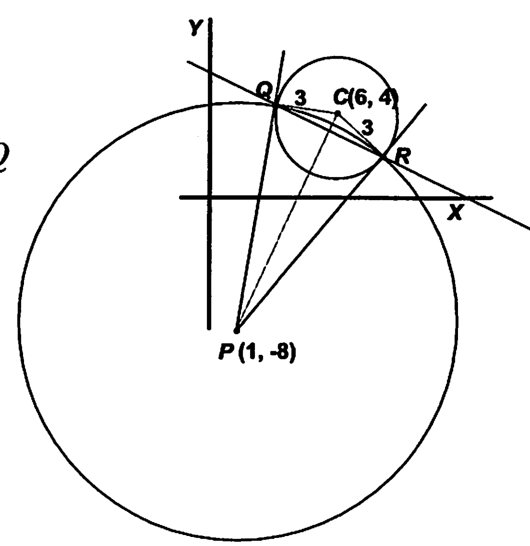
C) C has center at (6, 4) and radius 3.

$PC=\sqrt{(6-1)^2+(4-{}^-8)^2}=\sqrt{25+144}=13$

and $PR^2 = 169 - 9 = 160$.

Therefore, the circle with center at (1, −8) passing through Q and R has equation $(x-1)^2+(y+8)^2=160$

Since $Q(x_1, y_1)$ and $R(x_2, y_2)$ lie on both circles, the coordinates satisfy both equations and, consequently, satisfy *the difference of the two equations*. Subtracting the two equations, we have the equation of the line containing the common chord.



$$\begin{cases}(x-1)^2+(y+8)^2=160\\(x-6)^2+(y-4)^2=9\end{cases}$$

$$\Leftrightarrow\begin{cases}x^2+y^2-2x+16y=95\\ \underline{x^2+y^2-12x-8y=-43}\end{cases}$$

$$10x+24y=138 \Leftrightarrow \underline{\mathbf{5x+12y-69=0}}.$$

Note: It was *<u>not</u>* necessary to find the coordinates of either Q or R.
This was fortunate indeed, since the actual points of intersection would have been a royal pain to compute, namely, $\left(\frac{3\left(323\pm 48\sqrt{10}\right)}{169},\frac{4\left(142\mp 15\sqrt{10}\right)}{169}\right)$. And there's more. We haven't yet found the equation of the line passing through these two points, let alone simplified it!

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) _______________________________

B) _______________________________

C) _______________________________

A) Compute k, if $f(x) = 4^{3x+1}$ and $g(x) = k \cdot 8^{2x-3}$ are identical functions.

B) $x = k$ is a solution of $(\log_2 x)^2 = \log_2(x^2) + 8$
Compute <u>all</u> possible values of $\log_8 k$.

C) Compute $\log_x y$, if $\begin{cases} 4^{9x} - 3 \cdot 2^{3(4x+y)} + 3 \cdot 2^{6(x+y)} - 8^{3y} = 0 \\ x + 2y = 20 \end{cases}$

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 4

A) $4^{3x+1} = k \cdot 8^{2x-3} \Leftrightarrow 2^{6x+2} = k \cdot 2^{6x-9} \Rightarrow k = \dfrac{2^{6x+2}}{2^{6x-9}} = 2^{11} = \underline{\mathbf{2048}}$.

B) $(\log_2 x)^2 = \log_2(x^2) + 8 \Leftrightarrow (\log_2 x)^2 = 2\log_2 x + 8 \Leftrightarrow (\log_2 x - 4)(\log_2 x + 2) = 0 \Rightarrow \log_2 x = 4, -2$

$\Rightarrow x = k = 2^4, 2^{-2} = 16, \dfrac{1}{4}$.

$A = \log_8 k \Leftrightarrow k = 8^A = 2^{3A}$.

$2^{3A} = 16 = 2^4 \Rightarrow A = \underline{\dfrac{4}{3}}$.

$2^{3A} = 2^{-2} \Rightarrow A = \underline{-\dfrac{2}{3}}$.

C) $4^{9x} - 3 \cdot 2^{3(4x+y)} + 3 \cdot 2^{6(x+y)} - 8^{3y} = 0 \Leftrightarrow (2^{6x})^3 - 3 \cdot 2^{3(4x+y)} + 3 \cdot 2^{6(x+y)} - (2^{3y})^3 = 0$

Coefficients of 1, -3, 3, -1 suggest this might be the expansion of the cube of a difference, since $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$.

Examining the two middle terms, $\begin{cases} -3(2^{6x})^2(2^{3y}) = -3 \cdot 2^{12x} \cdot 2^{3y} = -3 \cdot 2^{3(4x+y)} \\ 3(2^{6x})(2^{3y})^2 = 3 \cdot 2^{6x} \cdot 2^{6y} = 3 \cdot 2^{6(x+y)} \end{cases}$.

This verifies that the first equation is the expansion of $(2^{6x} - 2^{3y})^3 = 0 \Rightarrow 2^{6x} = 2^{3y} \Rightarrow y = 2x$.

Substituting in the second equation, $x + 2y = 5x = 20 \Rightarrow (x, y) = (4, 8)$.

Thus, $N = \log_x y = \log_4 8 \Leftrightarrow 4^N = 8 \Leftrightarrow 2^{2N} = 2^3 \Rightarrow N = \underline{\mathbf{1.5}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

ANSWERS

A) ______________________________

B) ______________________________

C) (_______ , _______ , _______)

A) Given: y varies inversely as the cube of x, and w varies directly as the square of x. $w = 3$, when $(x, y) = (1, 4)$.

Compute w, when $y = \frac{1}{2}$.

B) A jet flew from Montreal to Reykjavik, Iceland, a distance of 2200 miles. On the return trip over the same route, the pilot increased the jet's speed by 110 mph and completed the trip in one less hour. Compute the <u>faster</u> speed of the jet.

C) A math team has at least 60 members. The number of members is the minimum number that allows the ratio of boys to girls to be 6 : 1. If one more girl joins the team, and n boys are dropped from the team, the ratio of girls to boys is 2 : 5. For the initial minimum number of members on the math team, if x new girls join the team and the same number of boys drop, the boy-to-girl ratio becomes 2 : 1. If this happens again next year (x new girls join the team and the same number of boys drop), the reduced boy-to-girl ratio becomes $a : b$. Compute the ordered triple (n, a, b).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017 SOLUTION KEY

Round 5

A) If $y=\frac{k_1}{x^3}$, $w=k_2x^2$ and $(x,y,w)=(1,4,3)$, then $k_1=4$ and $k_2=3$.

Substituting, if $y=\frac{1}{2}$, then $\frac{1}{2}=\frac{4}{x^3}\Rightarrow x^3=8\Rightarrow x=2$

$x=2\Rightarrow w=3(2)^2=\underline{\mathbf{12}}$.

B) The ratio $\frac{\text{Distance}}{\text{Time}}$ equals Rate (i.e. speed).

$R_{new}=\frac{2200}{t-1}=R_{old}+110=\frac{2200}{t}+110$

$\Rightarrow 2200t=2200(t-1)+110t(t-1)$

$\Rightarrow 110t^2-110t-2200=0\Leftrightarrow t^2-t-20=(t-5)(t+4)=0\Rightarrow t=5$

Therefore, $R_{new}=\frac{2200}{5-1}=\underline{\mathbf{550}}$ mph.

C) $\frac{G}{B}=\frac{k+1}{6k-n}=\frac{2}{5}\Rightarrow 5k+5=12k-2n\Rightarrow k=\frac{2n+5}{7}$ Think: slope $\frac{2}{7}$

$\Rightarrow (n,k)=(1,1),(8,3),(15,5),(22,7),(29,9),\ldots$

Since the math team, initially, had ($7k$) members (k girls and $6k$ boys), a minimum value of $k=9$ gives us 63 members ($k=8$ falls short at 56 members).

Check: After adding 1 girl and dropping $n=29$ boys, the ratio is $\frac{9+1}{54-29}=\frac{10}{25}=\frac{2}{5}$.

$\frac{54-x}{9+x}=\frac{2}{1}\Rightarrow 54-x=18+2x\Rightarrow 3x=36\Rightarrow x=12$.

Increase the girls and decrease the boys again, and we have

$\frac{B}{G}=\frac{54-12}{9+12}=\frac{42}{21}=2:1\Rightarrow(\text{next year})\frac{30}{33}\Rightarrow (n,a,b)=\underline{\mathbf{(29,10,11)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

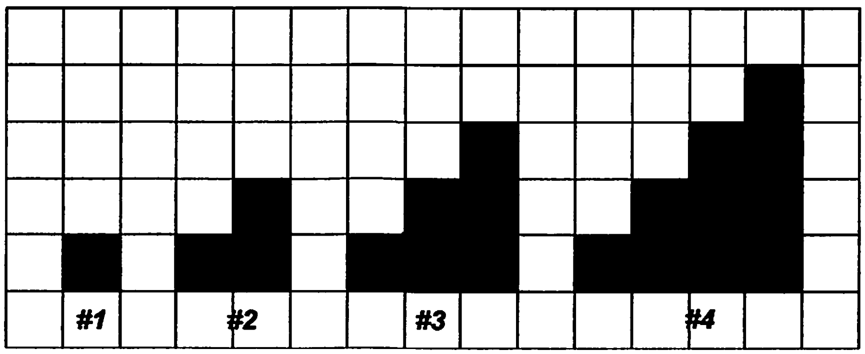
ANSWERS

A) ______________________

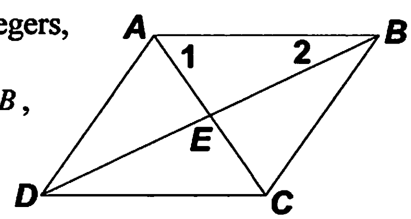
B) (__________ , __________)

C) ______________________

A) In the diagram below, each small square in the grid is one inch on a side. Assume the pattern continues. How many shapes will have a perimeter P such that $2000 \le P \le 2025$?



B) In rhombus $ABCD$, $m\angle 1 : m\angle 2 = a : b$, a simplified ratio of integers, where $a + b \le 15$ and $m\angle 2 < 45°$.
Let m and M denote the smallest and largest measures of $\angle DAB$, respectively.
Compute the ordered pair (m, M).

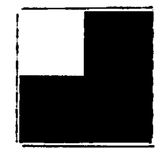


C) P and Q are regular polygons. P has 3 more sides and 135 more diagonals than Q.
The positive difference (in degrees) between the measure of an interior angle of P and an interior angle of Q is k. Compute k.

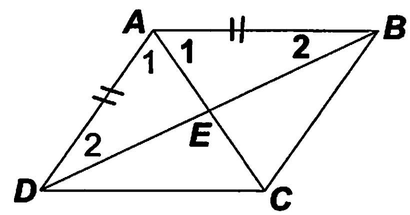
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017 SOLUTION KEY

Round 6

A) Notice the perimeter of any region and the "circumscribing" rectangle are the same. Thus, figures 1 …4 have perimeters 4, 8, 12, and 16, i.e. multiples of 4. Over the interval $2000 \le P \le 2025 \Leftrightarrow 2000 \le 4k \le 2025 \Rightarrow 500 \le k \le 506$. Thus, there are <u>7</u> multiples of 4, namely 2000, 2004, 2008, …, 2024.



B) Let $(m\angle 1, m\angle 2) = (ax, bx)$.
Since $\overline{AE} \perp \overline{BE}$, $m\angle 1 + m\angle 2 = 90°$.
$(a,b) = (14,1) \Rightarrow 15x = 90° \Rightarrow x = 6° \Rightarrow m\angle 1 = 14x = 84°$,
$m\angle 2 = 6° < 45° \Rightarrow m\angle DAB = 168°$ (max)
$(a,b) = (8,7) \Rightarrow 15x = 90° \Rightarrow x = 6° \Rightarrow m\angle 1 = 8x = 48°$,
$m\angle 2 = 42° < 45° \Rightarrow m\angle DAB = 96°$ (min)
Thus, $(m, M) = \underline{\mathbf{(96,168)}}$.



C) Assume P has $(n+3)$ sides and Q has n sides. Then:

P has $\dfrac{(n+3)n}{2}$ diagonals, while Q has $\dfrac{n(n-3)}{2}$ diagonals.

$$\frac{(n+3)n}{2} = \frac{n(n-3)}{2} + 135 \Rightarrow n^2 + 3n = n^2 - 3n + 270$$
$$\Rightarrow 6n = 270 \Rightarrow n = 45.$$

The difference in the interior angle measures is

$$180\left(\frac{(48-2)}{48} - \frac{(45-2)}{45}\right) = \overset{4}{\cancel{180}}\left(\frac{46\cdot 45 - 43\cdot 48}{\cancel{45}\cdot 48}\right) = \frac{2070-2064}{12} = \frac{1}{2} \Rightarrow k = \underline{\mathbf{\frac{1}{2}}} \text{ or } \underline{\mathbf{0.5}}.$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ______________________ D) (___ , ___ , ___ , ___ , ___)

B) ______________________ E) ______________________

C) ______________________ F) ______________________

A) Given: A box (i.e., a rectangular solid) with faces having areas of 180 square units, 240 square units and 144 square units. Let C be the center of the box and P and Q be consecutive vertices of a face. Compute the maximum value of $\cos(\angle PCQ)$.

B) $N = a_1a_2a_3a_4a_5$ is a 5-digit base 10 integer.
If $0 < a_1 < a_5 < a_2 < a_4 < a_3$, then N is called an increasing *spiral* number.
For example, 13542 and 13842 are increasing spiral numbers; 27895, 73015, and 14651 are not. How many natural numbers are increasing spiral numbers as described?

C) A rectangle with sides parallel to the coordinates axes has a pair of opposite vertices at the origin and $P(2k, k+1)$. For a minimum positive integer value of k, the distance from the origin to the lattice point in the interior of the rectangle furthest from the origin, namely, $Q(2k-1, k)$ is an integer. Consider circles $\begin{cases}(x,y) \mid x^2+y^2=k^2 \\ (x,y) \mid x^2+y^2=(2k+1)^2\end{cases}$.
How many lattice points inside the rectangle lie in the region between these two circles?

D) The function $f(x) = \log_B(2^x + C)$ has x-intercept at (–3, 0) and y-intercept at (0, 1).
Suppose $\log 2 = P$ and $\log 3 = Q$.
In simplified form, $f(4) = \dfrac{aP + bQ + c}{dP + eQ + f}$, where $a, d > 0$. Determine (a, b, c, d, e, f)

E) Since 2017 is a prime number, the quotient $\dfrac{2017}{n}$, where n is an integer and $2 \le n \le 666$ will always leave a nonzero remainder. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
Let S_1 be a subset of S for which the remainder of $\dfrac{2017}{n} \in S$.
Let S_2 be a subset of S for which the remainder of $\dfrac{2017}{n} \notin S$.
Let $N(S)$ denote the number of integers in set S.
Compute the ratio $N(S_1) : N(S_2)$.

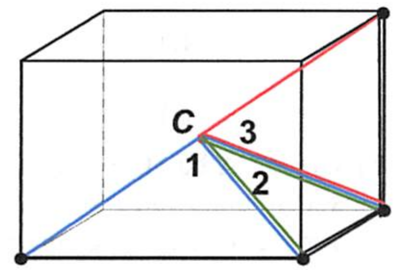
F) Compute the number of different triangles formed by the vertices of a convex pentagon and the interior intersection points of any four of its diagonals.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017 SOLUTION KEY

Team Round

A) Let the sides of the box be denoted *L*, *W*, and *H*. Then:

$$\begin{cases} LW = 180 \\ LH = 240 \Rightarrow (L,W,H) = \left(10\sqrt{3}, 6\sqrt{3}, 8\sqrt{3}\right) \\ WH = 144 \end{cases}$$



All 4 spatial diagonals (passing through the center) will have the same length, namely,

$$d^2 = \left(10\sqrt{3}\right)^2 + \left(8\sqrt{3}\right)^2 + \left(6\sqrt{3}\right)^2 = 300 + 192 + 108 = 600.$$

There are 3 cases. $\overline{PQ}$ could be the base of a front face, a side face, or a top face. Using the law of cosines, $PQ^2 = \left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 - 2\cdot\frac{d}{2}\cdot\frac{d}{2}\cdot\cos\theta \Leftrightarrow \cos\theta = \dfrac{2\cdot\frac{d^2}{4} - PQ^2}{2\cdot\frac{d^2}{4}} = \dfrac{300 - PQ^2}{300}$,

where $\theta = m\angle 1, m\angle 2$, or $m\angle 3$. Since the cosine is a decreasing function for $0 < \pi < 180°$, we want to pick the smallest possible value of θ. Since each triangle is isosceles and the legs are each $\frac{d}{2}$, the smallest θ occurs opposite the shortest edge of the rectangular solid.

Substituting for *PQ*, $\cos\theta = \dfrac{300 - \left(6\sqrt{3}\right)^2}{300} = \dfrac{192}{300} = \mathbf{\dfrac{16}{25}}$.

B) Given: $N = a_1a_2a_3a_4a_5$, where $0 < a_1 < a_5 < a_2 < a_4 < a_3$. Suppose the gaps add up to 4.
13542 is a spiral number produced by the gaps 1111.
The starting number may range from 1 to 5.
The additional 1111 spiral numbers are 24653, 35764, 46875 and 57986.

Total : 5

Suppose the gaps add up to 5. The starting number may range from 1 to 4.
The only possible gaps are 1112 (or some rearrangement).

$4 \cdot 4 = 16$

Gaps add up to 6. Starting number may be 1 to 3
1113 $\Rightarrow 3\cdot 4 = 12$ 1122 $\Rightarrow 3\cdot 6 = 18$
Gaps add up to 7. The starting number may be 1 or 2.
1114 $\Rightarrow 2\cdot 4 = 8$ 1123 $\Rightarrow 2\cdot 12 = 24$ 1222 $\Rightarrow 2\cdot 4 = 8$
Gaps add up to 8. Starting number must be 1.
1115 $\Rightarrow 4$ 1124 $\Rightarrow 12$ 1133 $\Rightarrow 6$ 1223 $\Rightarrow 12$ 2222 $\Rightarrow 1$
Total: 5 + 16 + 30 + 40 +35 = **126**.

MASSACHUSETTS MATHEMATICS LEAGUE
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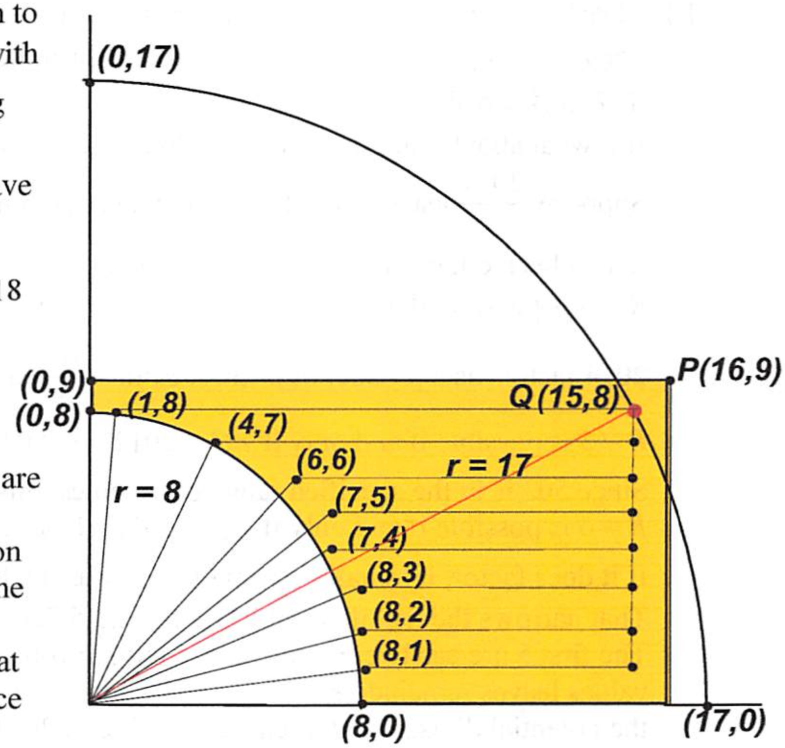
Team Round

C) Let d denote the distance from the origin to the point $Q(2k-1,k)$. Experimenting with common Pythagorean Triples or, finding integer ordered pairs which satisfy $d^2=(2k-1)^2+k^2=5k^2-4k+1$, we have $(d,k)=(17,8)$ [P.T.: 8-15-17]
($k=1...7 \Rightarrow d^2=2,13,34,65,106,157,218$ and none of these values are perfect squares.)
Thus, lattice points occur on each horizontal line from $y=1$ to $y=8$.
The rightmost points on each horizontal are always $(15, y)$.
The x-coordinate of the leftmost points on each horizontal must be chosen so that the sum of the squares of the coordinates is greater than $8^2=64$. From the diagram at the right, we see that the number of lattice points is
$14+12+10+9+9+8+8+8=$ **78**.



D) The graph of the log function usually does not have a y-intercept.

Substituting, $\begin{cases} 0=\log_B\left(2^{-3}+C\right) \\ 1=\log_B\left(2^0+C\right) \end{cases} \Rightarrow \begin{cases} B^0=C+\frac{1}{8} \\ B^1=1+C \end{cases} \Rightarrow (B,C)=\left(\frac{15}{8},\frac{7}{8}\right)$

Thus,

$$f(4)=\log_{\frac{15}{8}}\left(2^4+\frac{7}{8}\right)=\log_{\frac{15}{8}}\left(\frac{135}{8}\right)=\frac{\log_{10}\left(\frac{135}{8}\right)}{\log_{10}\left(\frac{15}{8}\right)}=\frac{\log_{10}135-\log_{10}8}{\log_{10}15-\log_{10}8}=\frac{3\log_{10}3+\log_{10}5-3\log_{10}2}{\log_{10}3+\log_{10}5-3\log_{10}2}$$

But $\log_{10}5=\log_{10}\left(\frac{10}{2}\right)=\log_{10}10-\log_{10}2=1-P$.

Substituting,

$$f(4)=\frac{3Q+(1-P)-3P}{Q+(1-P)-3P}=\frac{-4P+3Q+1}{-4P+Q+1}=\frac{4P-3Q-1}{4P-Q-1}\Rightarrow$$ **(4, –3, –1, 4, –1, –1)**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017 SOLUTION KEY

Team Round

E) Clearly, for $n = 2, 5,$ and 10, the remainders are easily calculated as 1, 2, and 7.
For $n = 11$, by "long division", we can confirm the remainder to be 4.
~~1~~ ~~2~~ 3 ~~4~~ 5 6 ~~7~~ 8 9
But what about other remainders? Which are possible and which are not?
Suppose $\frac{2017}{n}$ leaves a quotient of Q and a remainder of R. Then: $nQ + R = 2017$
Let's check out each of the other remainders.
$R = 3$ is possible if and only if $nQ = 2014 = 2(1007) = 2 \cdot 19 \cdot 53$
2014 factors as a product of primes. Both $\frac{2017}{19}$ and $\frac{2017}{53}$ will leave a remainder of 3.
$R = 5$ is possible if and only if $nQ = 2012 = 4(503)$.
Since 503 is in the specified range of n-values, this R-value is possible.
$R = 6$ is possible if and only if $nQ = 2011$. Does 2011 factor or is it prime?
If it does factor, its smallest prime factor must be less than or equal to $\sqrt{2011} < 45$.
That narrows the list of possibilities to $\{2,3,5,7,11,13,17,19,23,29,31,37,41,43\}$.
The first 5 are easily eliminated. It's left for you to verify that division by the remaining 9 values leaves remainders of 2, 11, 3, 16, 16, 2, 19, 8, and 39, respectively. Thus, since none of the potential divisors left a remainder of zero, 2011 is prime and $R = 6$ is impossible.
$R = 8$ is possible if and only if $nQ = 2009 = 7(287) = 7^2 \cdot 41$.
$n = 41$ produces the remainder of 8.
$R = 9$ is possible if and only if $nQ = 2008 = 8(251)$.
$n = 251$ produces the remainder of 9.
Thus, the only number R in S which cannot be a remainder of $\frac{2017}{n}$ is 6,
and the required ratio is **8 : 1**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2017 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) 1 B) 312, 306 C) $\frac{6\sqrt{13}}{65}$

Order is not important.

Round 2 Arithmetic/Elementary Number Theory

A) 14 B) 12 C) 14

Round 3 Coordinate Geometry of Lines and Circles

A) 12 B) 300π C) $5x + 12y - 69 = 0$

Round 4 Algebra 2: Log and Exponential Functions

A) 2048 B) $-\frac{2}{3}, \frac{4}{3}$ C) 1.5

Round 5 Algebra 1: Ratio, Proportion or Variation

A) 12 B) 550 C) (29, 10, 11)

Round 6 Plane Geometry: Polygons (no areas)

A) 7 B) $(96,168)$ C) $\frac{1}{2}$ or 0.5

Team Round

A) $\frac{16}{25}$ D) (4, –3, –1, 4, –1, –1)

B) 126 E) 8 : 1

C) 78 F) 20