MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017
ROUND 1 VOLUME & SURFACES

ANSWERS

A) ______________________________

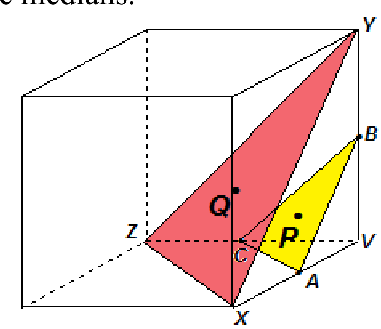
B) ______________________________

C) ______________________________

A) The height of a cylinder is 6 units. Compute the radius of the base, if the lateral surface area equals the area of one of its bases.

B) The volume of a sphere is $\frac{9\pi}{2}$ cubic units. Compute its surface area.

C) Given a cube with one vertex V and vertices X, Y, and Z, each of which is adjacent to V. The midpoints of edges $\overline{VX}$, $\overline{VY}$, and $\overline{VZ}$ are A, B, and C, respectively.
The volume of pyramid $ABCV$ is $\frac{4}{3}$.
If P is the center of ΔABC and Q is the center of ΔXYZ, compute PQ.
Assume the center of a triangle is the point of intersection of its three medians.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017 SOLUTION KEY

Round 1

A) Unzipping the cylinder along a line perpendicular to the bases and rolling out the lateral surface, we obtain a rectangle. Thus, the lateral surface area of a cylinder of radius r and height h is given by $2\pi rh$. Thus, $\pi r^2 = (2\pi r)6 \Leftrightarrow r = 2\cdot 6 = \underline{\mathbf{12}}$.

B) $V = \frac{4}{3}\pi r^3 = \frac{9\pi}{2} \Rightarrow r^3 = \frac{9}{2}\cdot\frac{3}{4} = \frac{27}{8} \Rightarrow r = \frac{3}{2}$

Substituting in $\text{SA} = 4\pi r^2$, we have $(4\pi)\cdot\frac{9}{4} = \underline{\mathbf{9\pi}}$.

C) Pyramid $ABCV$ is a corner of the given cube. Let $AV = m$. Consider ΔBAV the base.

$$V_{\text{pyramid}} = \frac{1}{3}B\cdot h \Rightarrow \begin{cases} V_{ABCV} = \frac{1}{3}\left(\frac{1}{2}m^2\right)\cdot m = \frac{1}{6}m^3 = \frac{4}{3} \Rightarrow m = 2 \Rightarrow (AV, AB) = \left(2, 2\sqrt{2}\right) \\ V_{XYZV} = \frac{1}{3}\left(\frac{1}{2}4^2\right)\cdot 4 = \frac{32}{3} \end{cases}.$$

ΔABC is an *equilateral* triangle with side-length $s = 2\sqrt{2}$ and area $\frac{s^2\sqrt{3}}{4} = 2\sqrt{3}$.

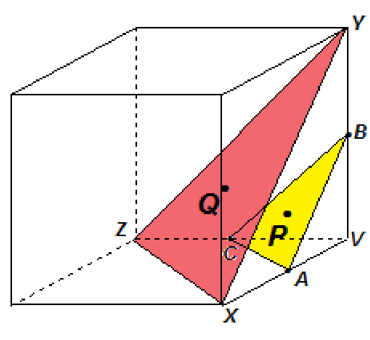
ΔXYZ is also equilateral with a side-length of $4\sqrt{2}$ and area $8\sqrt{3}$.

Now, re-orient your point of view so that ΔABC is the base of pyramid $ABCV$ and ΔXYZ is the base of pyramid $XYZV$. V, P, Q, and the vertex of the cube opposite V are collinear. All these points lie on a (space) diagonal of the cube. From this point of view, $\overline{VP}$ and $\overline{VQ}$ are heights of their respective pyramids.

$$\frac{1}{3}\cdot VP\cdot\left(2\sqrt{3}\right) = \frac{4}{3} \Rightarrow VP = \frac{2\sqrt{3}}{3}$$

$$\frac{1}{3}\cdot VQ\cdot\left(8\sqrt{3}\right) = \frac{32}{3} \Rightarrow VQ = \frac{4\sqrt{3}}{3}$$

Thus, $PQ = VQ - VP = \underline{\mathbf{\frac{2\sqrt{3}}{3}}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

ANSWERS

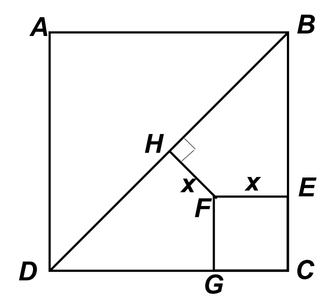
A) ______________________________

B) ______________________________

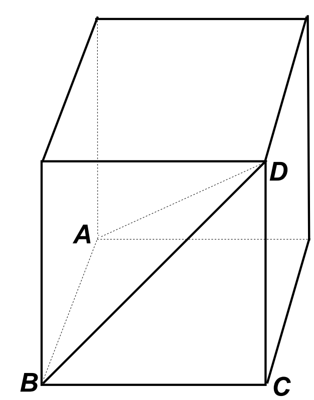
C) ______________________________

A) The legs of a right triangle have lengths 60 and 63 and the length of the hypotenuse is an integer. Compute the perimeter of this triangle.

B) Given: $ABCD$ is a square and $AB = \sqrt{2}$
Compute the area of square $CEFG$.



C) In the box shown at the right, $AB : BC = 3 : 4$, $BC : CD = 1 : 3$, and diagonal $AD = 26$. Compute the area of <u>right</u> ΔABD.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017 SOLUTION KEY

Round 2

A) Solution #1: Special right triangles

Let (a,b,c) denote the three sides of the triangle, where c is the hypotenuse.

Noticing a common factor of 3, we have $(a,b,c)=3(20,21,x)$.

Since $(20,21,29)$ is a special right triangle, we have $c=3\cdot 29=87$ and the perimeter is **<u>210</u>**.

Solution #2: Trial and Error

$60^2+63^2=3600+3969=7569=c^2$, where c denotes the length of hypotenuse.
The length of the hypotenuse must lie between 64 and 125, applying the triangle inequality and recognizing that the hypotenuse is the longest side. Even more restrictive, since $80^2=6400$ and $90^2=8100$, we know the length of the hypotenuse is between 80 and 90. Only 83^2 and 87^2 have a units digit of 9 and the winner is 87, producing a perimeter of 210.

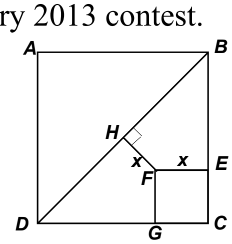
Solution #3 Brute Force (Extracting a square root)

Starting at the decimal point, group the digits in pairs. $7569 \Rightarrow 7569. \Rightarrow 75\ 69$
Determine the largest integer n whose square does not exceed the leftmost pair.
Square, subtract and bring down the next pair, $75-8^2=11 \Rightarrow 1169$. Call this value T.
Double n and attach a new units digit d such that the product of $(10(2n)+d)d \le T$. Thus, it is required that $(160+d)d \le 1169$. If d is taken to be 7, we have $167(7)=1169$ and we have the square root.
The steps are summarized in the diagram at the right.
Try extracting the square roots of the following perfect squares: 729 2401 19321
Try computing $\sqrt{2}$.
Think of 2 as 02. 00 00 00 ...

```
          8  7
       )75  69
        64
  167   11  69
        11  69
             0
```

The algorithm for computing square root was previously discussed in the January 2013 contest. It is included at the end of the solution key for those who might be interested.

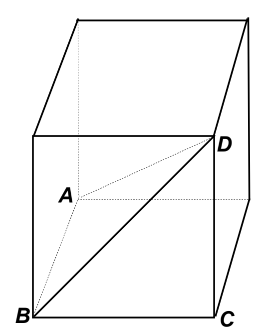
B) H and F must lie on the diagonal $\overline{AC}$



$AB=\sqrt{2} \Rightarrow BD=AC=2 \Rightarrow CH=1$

Thus, $x+x\sqrt{2}=1 \Rightarrow x=\dfrac{1}{\sqrt{2}+1}=\sqrt{2}-1 \Rightarrow$ area $=\left(\sqrt{2}-1\right)^2=\underline{\mathbf{3-2\sqrt{2}}}$.

C) Let $(AB,BC)=(3x,4x)$. Then, using the Pythagorean Theorem, $AC=5x$.



$DC:BC=3:1 \Rightarrow DC=12x$

$$AC^2+DC^2=AD^2 \Rightarrow (5x)^2+(12x)^2=26^2 \Rightarrow x^2=\frac{26^2}{13^2}=4 \Rightarrow x=2$$

Since ΔABD is a right triangle,

$$\text{Area}(\Delta ABD)=\frac{1}{2}(AB)(BD)=\frac{1}{2}(6)\left(\sqrt{8^2+24^2}\right)=3(8)\sqrt{1+9}=\underline{\mathbf{24\sqrt{10}}}.$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) ______________________________

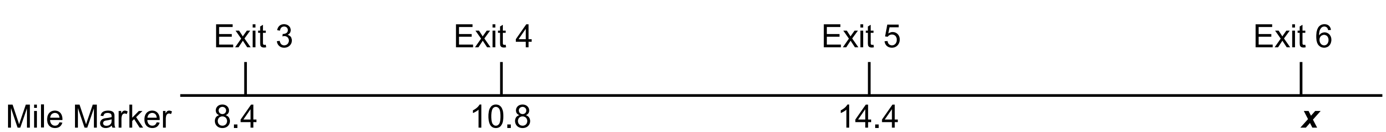
B) ______________________________

C) ______________________________

A) My lucky number is 6 more than your lucky number. Adding 1 to my lucky number and then dividing by 2 produces your lucky number. What is my lucky number?

B) Given: $\frac{x-a}{4} = 2(b-3x)$, where a and b are positive integer constants.
If $x = 1$ is a solution of this equation, compute the <u>largest</u> possible value of $a + b$.

C) Compute the mile marker at Exit 6, if my average speed between exits 3 and 6 was 70 mph.
Between exits 3 and 4, I was travelling at 60 mph.
I passed exit 4 at 8:57 PM and exit 5 at 9 PM.
It took me 3 minutes and 36 seconds to travel from exit 5 to exit 6.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017 SOLUTION KEY

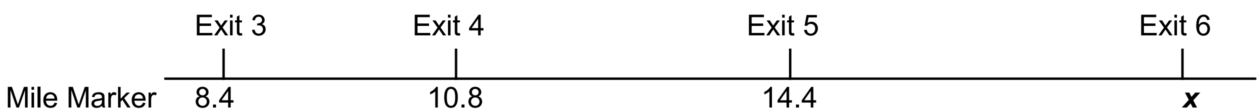
Round 3

A) Let x denote my lucky number (and $x - 6$ yours). Then:

$$\frac{x+1}{2} = x - 6 \Rightarrow x + 1 = 2x - 12 \Rightarrow x = \underline{\mathbf{13}}.$$

B) $x = 1 \Rightarrow \dfrac{1-a}{4} = 2(b-3) \Rightarrow 1 - a = 8b - 24 \Rightarrow a + 8b = 25$

The possible ordered pairs (a, b) are (1,3), (9, 2) and (17, 1).
Thus, the largest value of $a + b$ is **<u>18</u>**.



C) Given: 60 mph between exits 3 and 4, passed exit 4 at 8:57 PM, exit 5 at 9 PM, and a travel time of 3 minutes and 36 seconds from exit 5 to exit 6
Note: From exit 4 to exit 5 took 3 minutes = 1/20$^{\text{th}}$ hour.

$$Rate = \frac{Distance(mi)}{Time(hr)} \Rightarrow 70 = \frac{x - 8.4}{\frac{2.4}{60} + \frac{1}{20} + \frac{3.6}{60}} = \frac{x-8.4}{0.04 + 0.05 + .06} = \frac{x-8.4}{0.15}$$

Cross multiplying, we have $10.5 = x - 8.4 \ = \underline{\mathbf{18.9}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) If the digits of a two-digit prime number are reversed, the new number is also prime.
Let N be the number of primes with this property.
Let S be the sum of all primes with this property.
Compute $\left\lfloor \frac{S}{N} \right\rfloor$, where $\lfloor x \rfloor$ denotes the <u>floor</u> function (the largest integer less than or equal to x).
Note: 11 is not considered, since reversing the digits does not produce a "new" number.

B) Given: $\frac{3x}{(x-4)^2} = \frac{A}{x-4} + \frac{B}{(x-4)^2}$
Compute $A - B$.

C) Consider the following rules for adding fractions:
#1: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ (correct) #2: $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$ (incorrect)

Both rules are applied to the sum $\frac{5}{12} + \frac{x}{24}$.
For $x = K$, the incorrect rule gives the correct answer.
For $x = J$, the absolute value of the difference of the results given by each rule is $\frac{1}{2}$.
Compute <u>all</u> possible ordered pairs (K, J).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017 SOLUTION KEY

Round 4

A) The only possible units digits of a 2-digit prime are 1, 3, 7 and 9.

$$S=(13+31)+(17+71)+(37+73)+(79+97)=44+88+110+176=418$$

$$\left\lfloor \frac{S}{N} \right\rfloor = \left\lfloor \frac{418}{8} \right\rfloor = \left\lfloor 52^{+} \right\rfloor = \underline{\mathbf{52}}.$$

B) $$\frac{3x}{(x-4)^2}=\frac{A}{x-4}+\frac{B}{(x-4)^2}=\frac{A(x-4)+B}{(x-4)^2}$$

Thus, $3x = A(x-4)+B$ for <u>all</u> $x \neq 4$.
$x=1 \Rightarrow 3=-3A+B$.
$x=2 \Rightarrow 6=-2A+B \Rightarrow A=3$ and $B=12 \Rightarrow A-B=\underline{\mathbf{-9}}$.

C) According to rule #1 (correct): $\dfrac{5}{12}+\dfrac{x}{24}=\dfrac{120+12x}{12\cdot 24}=\dfrac{10+x}{24}$

According to rule #2 (incorrect): $\dfrac{5}{12}+\dfrac{x}{24}=\dfrac{5+x}{36}$

For $x=K$, $\dfrac{10+K}{24}=\dfrac{5+K}{36} \Leftrightarrow \dfrac{10+K}{2}=\dfrac{5+K}{3} \Rightarrow 30+3K=10+2K \Rightarrow K=-20$

For $x=J$, $\left|\dfrac{10+J}{24}-\dfrac{5+J}{36}\right|=\dfrac{1}{2} \Leftrightarrow \left|3(10+J)-2(5+J)\right|=36 \Leftrightarrow |20+J|=36$

$\Rightarrow 20+J=\pm 36 \Rightarrow J=16,\,-56$.

Thus, $(K,J)=\underline{\mathbf{(-20,16),\ (-20,-56)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) How many integer multiples of 3 satisfy the inequality $|x-5| \le 2017$?

B) Solve for x. $|3x+2| > |2x+1|$

C) The graphs of $A = \{(x,y) : |x| + |y| \le 6\}$ and $B = \{(x,y) : |x-y| \ge c\}$ overlap. Compute c, if the area of the overlapping region is 48.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017 SOLUTION KEY

Round 5

A) $|x-5| \le 2017 \Leftrightarrow -2017 \le x-5 \le +2017 \Leftrightarrow -2012 \le x \le 2022$

Call $x = 3k$. Then: $-670\frac{2}{3} \le k \le 674 \Rightarrow -670 \le k \le 674$

Thus, there are $674-(-670)+1 = \underline{\mathbf{1345}}$ multiples of 3.

B) $|3x+2| > |2x+1| \Leftrightarrow \sqrt{(3x+2)^2} > \sqrt{(2x+1)^2}$

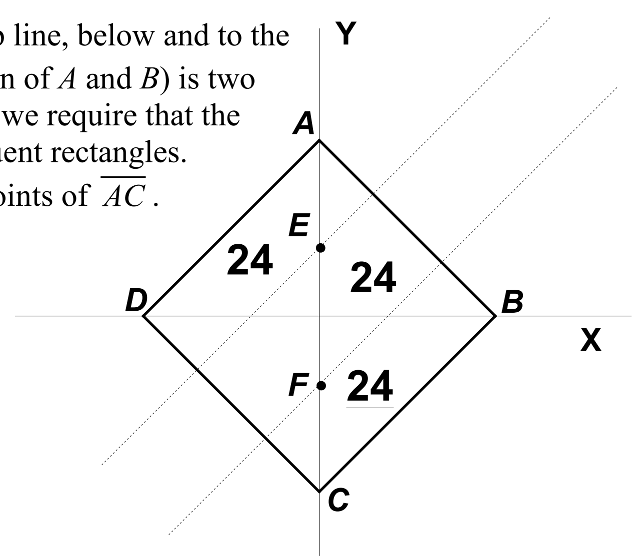
Since both sides are nonnegative, we can square both sides to remove the radicals and square out the binomials. $9x^2+12x+4 > 4x^2+4x+1 \Leftrightarrow 5x^2+8x+3 > 0 \Leftrightarrow (5x+3)(x+1) > 0$

The critical values are –1 and –3/5. For $x < -1$, both factors are negative.
As each critical value is passed, one more factor becomes positive.

Therefore, the product is positive for $\underline{\boldsymbol{x < -1 \text{ or } x > -\frac{3}{5}}}$.

Alternate solution (Annalisa Peterson - Mt. Alvernia):
$2x+1$ is either $2x+1$ or $-2x-1$
If $3x+2 = 2x+1$, then $x = -1$.
If $3x+2 = -2x-1$, then $x = -\frac{3}{5}$.
These critical values divide the number line into three regions.
Substituting any typical value from each region in the original inequality,
we see the inequality is satisfied if $\underline{\boldsymbol{x < -1 \text{ or } x > -\frac{3}{5}}}$.

C) A is a square (diamond) with vertices at $(\pm 6, 0)$ and $(0, \pm 6)$. Its area may be computed as half the product of its diagonals, i.e., $\frac{1}{2}(12^2) = 72$. B is the union of two half-planes outside the parallel lines $x - y = \pm c$ (above and to the left of the top line, below and to the right of the bottom line). Thus, $A \cap B$ (the union of A and B) is two congruent rectangles with a total area of 48 and we require that the parallel lines divide the square into three congruent rectangles.
This requires that point E and F are trisection points of $\overline{AC}$.
$\frac{12}{3} = 4 \Rightarrow c = 6 - 4 = \underline{\mathbf{2}}$



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A) (_______ , _______ , _______)

B) _______________________________

C) _______________________________

A) Great Britain formerly used coins with the following denominations: pence, shillings and pounds.
12 pence (d) = 1 shilling (s)
20 shillings (s) = 1 pound (£)
I bought 2 articles and paid £2 12s 10d for the first article and £1 15s 8d for the second.
I gave the clerk a 5-pound note.
My change will be £A Bs Cd.
Compute the ordered triple (A, B, C).

B) The floor function $\lfloor x \rfloor$ (or the greatest integer function $[x]$) is the *largest* integer not greater than x.
The ceiling function $\lceil x \rceil$ is the *smallest* integer not less than x.
Compute $20\left\lfloor -\frac{22}{7} \right\rfloor - 15\left\lceil \frac{19}{9} \right\rceil - 10\left\lfloor \frac{16}{11} \right\rfloor$.

C) Let d_Δ denote the middle digit of the natural number d. Compute *all* possible values of d_Δ, if d is the square of a 3-digit natural number whose units digit is 5 and for which $d < 40,000$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017 SOLUTION KEY

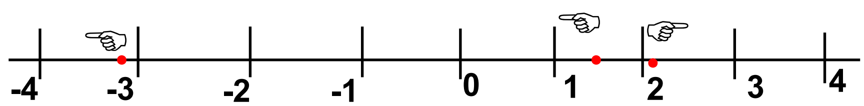
Round 6

A)

<u>Adding</u>			**<u>Subtracting</u>**			
pounds	shillings	Pence	5 Pounds =	4 pounds	19 shillings	12 pence
2	12	10		4	8	6
1	15	8		0	11	6
3	27	18				
4	8	6				

Thus, $(A, B, C) = \underline{\mathbf{(0, 11, 6)}}$.

B)



$$20\left\lfloor -\frac{22}{7} \right\rfloor - 15\left\lceil \frac{19}{9} \right\rceil - 10\left[\frac{16}{11} \right] = 20\left\lfloor -3\frac{1}{7} \right\rfloor - 15\left\lceil 2\frac{1}{9} \right\rceil - 10\left[1\frac{5}{11} \right] = 20(-4) - 15(3) - 10(1) = \underline{\mathbf{-135}}$$

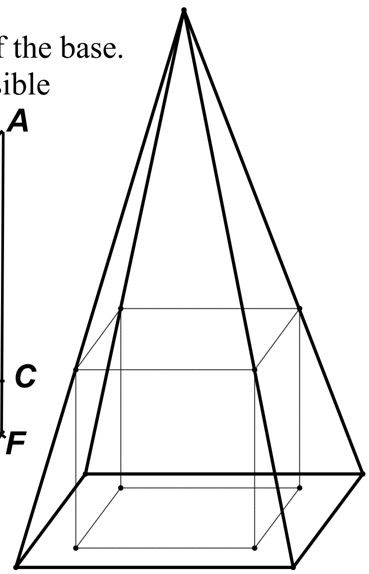
C) The square of any number ending in 5 will end in 25.
Since $(205)^2 > 40{,}000$, we must evaluate $105^2, 115^2, \ldots, 195^2$ and determine the middle digit.
Brute force would be painful, but if we recognize that
$(\underline{x}\,\underline{5})^2 = (10x+5)^2 = 100x^2 + 100x + 25 = 100(x)(x+1) + 25$, squaring multiples of 5 from 105 through 195 boils down to evaluating ten products $10 \cdot 11, 11 \cdot 12, 12 \cdot 13, \ldots, 19 \cdot 20$ and appending 25 as the rightmost two digits.
11**<u>0</u>**25, 13**<u>2</u>**25, 15**6**25, 18**<u>2</u>**25, 21**<u>0</u>**25, 24**<u>0</u>**25, 27**<u>2</u>**25, 30**<u>6</u>**25, 34**<u>2</u>**25, 38**<u>0</u>**25
Thus, the values of d_Δ are **<u>0</u>**, **<u>2</u>**, and **<u>6</u>** only.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017
ROUND 7 TEAM QUESTIONS
ANSWERS

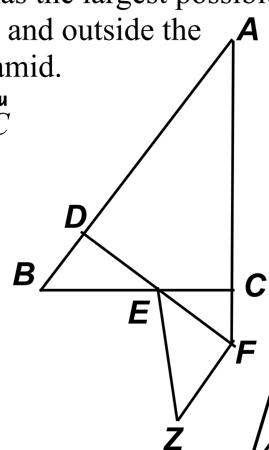
A) ___________________________ D) (___ , ___ , ___ , ___ , ___ , ___)

B) ___________________________ E) ___________________________

C) ___________________________ F) ___________________________



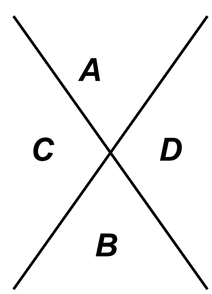
A) The height of a pyramid with a square base is twice the length of a side of the base. A cube with one face lying in the base of the pyramid has the largest possible volume. If the volume of the region inside the pyramid and outside the cube is 640, compute the lateral surface area of the pyramid.



B) Given: $AC = 4, BC = 3, BD = 1.2,\ EZ = 2,\ E = \overrightarrow{DF} \cap \overrightarrow{BC}$
$\overline{AC} \perp \overline{BC},\ \overline{DF} \perp \overline{AB},\ \overline{ZF} \perp \overline{FE}$
Compute FZ.

C) Points P, Q, R, S, and T are collinear and lie on line L in the given order (from left to right).
$PQ : QR : RS : ST = 1 : 2 : 3 : 4$
The following coordinates are given: $P(a, 1)$, $Q(2b, b)$, $R(34, c)$, $S(d, d)$. Compute the coordinates of point T.

D) Casting out 9s is a quick way to check for an error in a sum, product, or difference. Let A be the digit sum mod 9 of the first factor.
Let B be the digit sum mod 9 of the second factor. Let $C = AB$ mod 9
Let D be the digit sum of the given (possibly incorrect) product mod 9.
If $C \neq D$, then the answer is *definitely* wrong.
If $C = D$, then the answer is "probably" correct.
Suppose the product 784613 x 5036 is given as 3951311268.
One of the <u>even</u> digits in the answer is incorrect. Compute (A, B, C, D, w, c), where w denotes the incorrect digit and c the correct replacement.
Note: The mod operator returns the remainder of division by the specified divisor.
For example, 35 mod 9 = 8, since the quotient 35/9 returns a remainder of 8.



E) Points $P(-12, 32)$ and $Q(8, 28)$ lie on the graph of $F = \{(x, y) \mid y = |x+8| + |x+3| + |x-7|\}$.
Define $|PQ\rangle$ to be the distance from P to Q <u>*along the graph of F*</u>. If $|PQ\rangle = 5k$, compute k.

F) A 3-digit base 10 positive integer has/is
- a units digit that is prime
- a tens digit that is a non-prime multiple of 3
- a hundreds digit which is a composite factor of 8
- a multiple of at least one of $\{3, 4, 5, 11\}$

How many distinct numbers satisfy these conditions?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017 SOLUTION KEY

Team Round

A) Let h denote the height of the pyramid ($\overline{VD}$) and s the length of the side of the square base ($\overline{PQ}$). Then: $(h,s)=(2k,k)$. Let x denote the length of the edge of the cube.

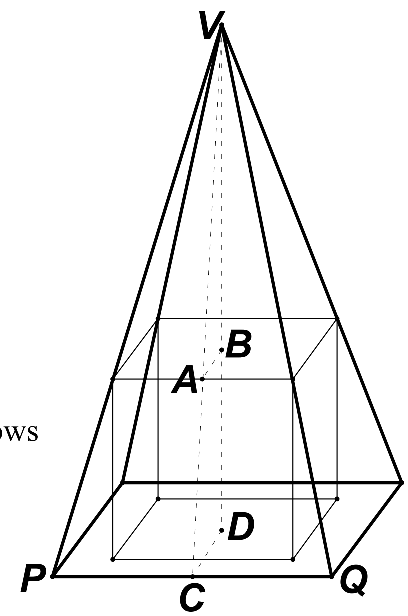
$$\Delta VAB \sim \Delta VCD \Rightarrow \frac{VB}{VD}=\frac{AB}{CD} \Leftrightarrow \frac{2k-x}{2k}=\frac{x/2}{k/2}=\frac{x}{k}$$

Cross multiplying, $2k^2-kx=2kx \Rightarrow x=\frac{2}{3}k$.

The required volume is $\frac{1}{3}s^2h-x^3=\frac{1}{3}k^2(2k)-\left(\frac{2}{3}k\right)^3=640$.

$$\Leftrightarrow\left(\frac{2}{3}-\frac{8}{27}\right)k^3=640 \Leftrightarrow \frac{10}{27}k^3=640 \Leftrightarrow k^3=64\cdot 27 \Rightarrow k=4\cdot 3=12$$

$$\Rightarrow (VD,CD)=(24,6) \Rightarrow VC=\sqrt{24^2+6^2}=\sqrt{6^2\left(4^2+1\right)}=6\sqrt{17}$$



Since C is the midpoint of the base of isosceles triangle VPQ, it follows that $\overline{VC}\perp\overline{PQ}$. Therefore, the lateral surface area is

$4\left(\frac{1}{2}\cdot 6\sqrt{17}\cdot 12\right)=\underline{\mathbf{144\sqrt{17}}}$.

B) $AB=5$. Let $BD=x$ (to avoid cumbersome arithmetic).

$$\Delta ABC \sim \Delta EBD \Rightarrow \frac{AB}{EB}=\frac{BC}{BD} \Rightarrow \frac{5}{EB}=\frac{3}{x} \Rightarrow EB=\frac{5}{3}x \Rightarrow EC=3-\frac{5}{3}x=\frac{9-5x}{3}$$

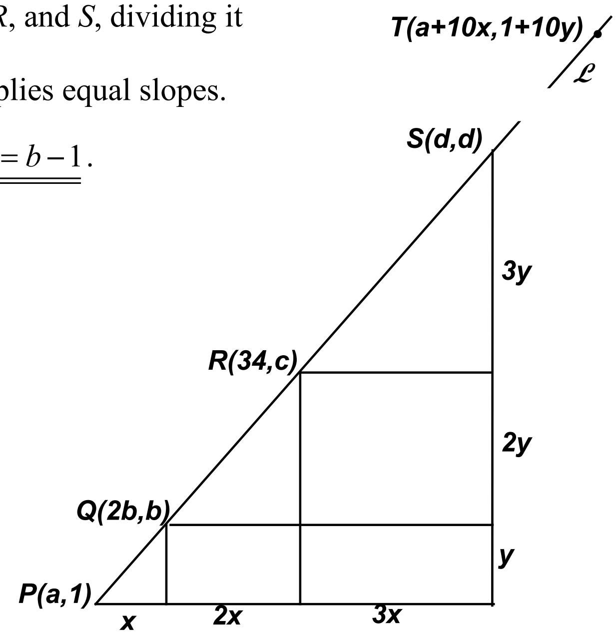
$$\Delta ABC \sim \Delta EFC \Rightarrow \frac{AB}{EF}=\frac{AC}{EC} \Rightarrow \frac{5}{EF}=\frac{4}{\frac{9-5x}{3}} \Rightarrow EF=\frac{5(9-5x)}{12}$$

Substituting for x and, using the Pythagorean Theorem on ΔFEZ , we have

$$EF=\frac{5}{4} \Rightarrow FZ=\sqrt{2^2-\left(\frac{5}{4}\right)^2}=\sqrt{\frac{64-25}{16}}=\underline{\mathbf{\frac{\sqrt{39}}{4}}}\ .$$

Team Round - continued

C) Perpendiculars which divide the horizontal and vertical segments into 1 : 2 : 3 ratio intersect on line *L* at points *P*, *Q*, *R*, and *S*, dividing it into the required ratio.
Referring to the diagram at the right, collinearity implies equal slopes.



$\boxed{\dfrac{b-1}{2b-a}=\dfrac{y}{x}}$ $P >> Q: a+x=2b \Rightarrow x=2b-a \Rightarrow \underline{\underline{y=b-1}}$.

Note: R is the midpoint of $\overline{PS}$.

$\therefore\ \dfrac{a+d}{2}=34 \Rightarrow d=68-a$, $\dfrac{1+d}{2}=c \Rightarrow d=2c-1$

Equating, $68-a=2c-1 \Rightarrow c=\dfrac{69-a}{2}$.

$P >> R: a+3x=34 \Rightarrow a+3(2b-a)=6b-2a$

$\Rightarrow b=\dfrac{17+a}{3}$

Substituting in the underlined expression,

$y=\dfrac{17+a}{3}-1=\dfrac{14+a}{3}$

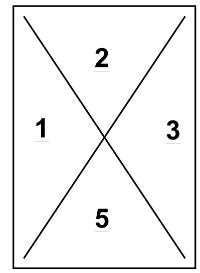
$P >> S:\ d=1+6y \Rightarrow 68-a=1+6\left(\dfrac{14+a}{3}\right) \Rightarrow 68-a=1+(28+2a) \Rightarrow 3a=39 \Rightarrow a=13$

Substituting, we have $d=55,\ b=10,\ c=28,\ x=7,\ y=9$

Finally, $T(a+10x,1+10y)=(13+70,\ 1+90)=\underline{\mathbf{(83,91)}}$

Team Round - continued

D) $A = (7+8+4+5+1+3) \text{ mod } 9 = 29 \text{ mod } 9 = 2,$
$B = (5+0+3+6) \text{ mod } 9 = 14 \text{ mod } 9 = 5,$
$C = (2 \cdot 5) \text{ mod } 9 = 1,$
$D = (3+9+5+1+3+1+1+2+6+8) \text{ mod } 9 = 39 \text{ mod } 9 = 3$



We must reduce the value of D by 2. Clearly, the units digit of 8 is correct, so either the 2 should have been 0 or the 6 should have been 4.
Of course, computing each sum was not necessary.
As soon as running total equals or exceeds 9, throw a 9 away; hence, the name "**casting out 9s**".
For example, processing 784613 might proceed either as

$7\cancel{8}4\cancel{6}\cancel{1}\cancel{3} \Rightarrow 7+4=11 \Rightarrow 2$ or

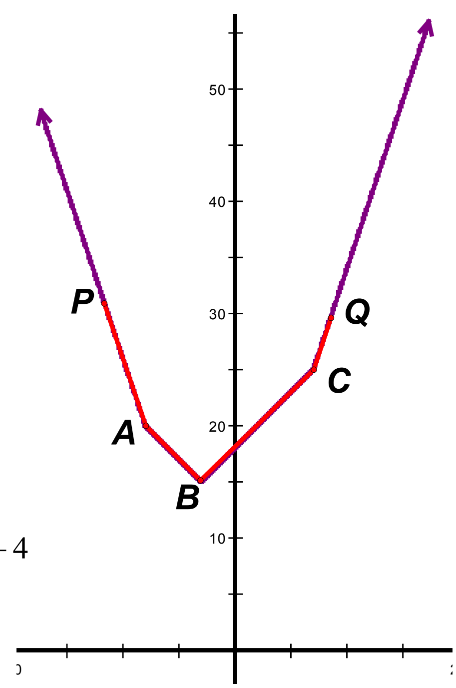
$(78)\,4\,6\,1\,3 \Rightarrow (6\,4)\,6\,1\,3$
$\Rightarrow (1\,6)\,1\,3 \Rightarrow (7\,1)3 \Rightarrow (8\,3)$
$\Rightarrow (11) \Rightarrow 2$

…	6	1	3
X	0	3	6
3	6	7	8
	3	9	0
	<u>0</u>	6	8

The table at the right examines the least significant digits of the product and shows that the 6 is correct and even verifies that replacing 2 with 0 fixes the error in the product.
Thus, $(A, B, C, D, w, c) =$ **<u>(2, 5, 1, 3, 2, 0)</u>**.

E) F is a continuous function over the reals with critical points at $x = -8, -3,$ and 7.
Thus, the graph passes through 3 noncollinear points A, B, and C.
Direct substation in the given equation gives us
$A(-8,20)$, $B(-3,15)$, and $C(7,25)$.

$$|PQ\rangle = PA + AB + BC + CQ = 4\sqrt{10} + 5\sqrt{2} + 10\sqrt{2} + \sqrt{10}$$
$$= 5\left(\underline{\mathbf{3\sqrt{2} + \sqrt{10}}}\right)$$



<u>FYI</u>: The graph of the piecewise function F is shown at the right.
For $-8 \le x \le 7$, we can simply "connect the dots".
For $x < -8$, the equivalent equation is
$y = (-x-8) + (-x-3) + (7-x) = -3x - 4$ which passes through
A and $P(-12,32)$. For $x > 7$, the equivalent equation is $y = 3x + 4$
which passes through C and $Q(8,28)$.

Team Round - continued

F) Let $N = \underline{H}\ \underline{T}\ \underline{U}$.
$U \in \{2,3,5,7\}$, $T \in \{0,6,9\}$ and $H \in \{4,8\}$
The rules:
N is divisible by
3 if and only if $H+T+U$ is a multiple of 3.
4 if and only if $10T+U$ is a multiple of 4.
5 if and only if $U = 0$ or 5.
11 if and only if $H+U-T$ is a multiple of 11.
Duplicates are crossed out
Divisible by 3: 402, 405, 462, 465, 492, 495, 807, 867, 897
Divisible by 4: ~~492~~, 892
Divisible by 5: ~~405~~, ~~465~~, ~~495~~, 805, 865, 895
Divisible by 11: 407, ~~462~~, ~~495~~, 803
Thus, there are **15** distinct numbers satisfying the given conditions.

Algorithm for Extracting Square Root sans Calculator

An example: *Determine the best two-decimal place approximation of* $\sqrt{8.15}$.

Group digits to the left and to the right of the decimal point into blocks of two.
Since we want accuracy to two decimal places, we write 8.15 as 08.15 00 00
The third decimal place will tell us if we need to round up.

The first digit is the largest N for which $N^2 \le$ leftmost twosome. $\quad N^2 \le 08 \Rightarrow N = 2$
Square N, subtract, and bring down the next twosome. Call this value X. $\quad X = 415$
Double the current approximation (2) and write this value (4) in the space at the left
Let d denote the next digit in the approximation.
We want $(4\,d)\cdot d$ to be less than or equal to X, i.e. forty-something times something $\le$ 415
$(48)\cdot 8 = 384 < 415$, but $(49)9 = 441 > 415$, so the next digit is 8.

This is summarized in the following template:

$$\begin{array}{ll} & \quad 2.d \\ & \sqrt{08.150000} \\ & \quad \underline{4} \\ 4\underline{d} & \quad 415 \\ & \quad \underline{384} \\ & \boxed{d = 8} \end{array}$$

Continue repeating these steps until the required number of decimal places have been determined.
Double the current approximation. Determine the next digit, i.e., largest d for which $(\ldots\, d)d \le X$.
Multiply / Subtract / Bring down the next twosome

The devil is in the details which are shown in the diagrams below:

$$\begin{array}{ll} & \quad 2.\ 8\ d \\ & \sqrt{08.150000} \\ & \quad \underline{4} \\ 48 & \quad 415 \\ & \quad \underline{384} \\ 56\underline{d} & \qquad 3100 \\ (565\cdot 5 = 2825 < 3100) & \\ (566\cdot 6 = 4396 > 3100) & \\ & \boxed{d = 5} \end{array}$$

$$\begin{array}{ll} & \quad 2.\ 8\ \ 5\ \ d \\ & \sqrt{08.150000} \\ & \quad \underline{4} \\ 48 & \quad 415 \\ & \quad \underline{384} \\ 565 & \qquad 3100 \\ & \qquad \underline{2825} \\ 570\underline{d} & \qquad\quad 27500 \\ (5704\cdot 4 = 22816) & \\ (5705\cdot 5 > 27500) & \\ & \boxed{d = 4} \end{array}$$

In practice, the calculations to determine d are not shown and all the computations are combined into a single template.
Thus, rounded to two decimal places, $\sqrt{8.15} = 2.85$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2017 ANSWERS

Round 1 Geometry Volumes and Surfaces

A) 12 B) 9π C) $\dfrac{2\sqrt{3}}{3}$

Round 2 Pythagorean Relations

A) 210 B) $3-2\sqrt{2}$ C) $24\sqrt{10}$

Round 3 Linear Equations

A) 13 B) 18 C) 18.9

Round 4 Fraction & Mixed numbers

A) 52 B) -9 C) $(-20,-56),(-20,16)$

Round 5 Absolute value & Inequalities

A) 1345 B) $x<-1$ or $x>-\dfrac{3}{5}$ C) 2

Round 6 Evaluations

A) $(0,11,6)$ B) –135 C) 0, 2, and 6 (in any order)

Team Round

A) $144\sqrt{17}$ D) (2, 5, 1, 3, 2, 0)

B) $\dfrac{\sqrt{39}}{4}$ E) $3\sqrt{2}+\sqrt{10}$

C) (83, 91) F) 15