MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Given: $f(x)=\dfrac{8-2x}{3}$ and $g(x)=f(2x)+1$

The graph of $y=g(x)$ intersects the x-axis at $(h,0)$. Compute h.

B) Given: $f(x)=\dfrac{k}{x+2}$

Compute the <u>nonzero</u> value(s) of k for which $f(2)=f^{-1}(4)\cdot f(4)$.

C) A line with slope 9 intersects a line tangent to $y=f(x)=x^3-6x^2-4x+24$ at $(2,0)$.

Compute the coordinates of the two other points where this line intersects $y=f(x)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017 SOLUTION KEY

Round 1

A) $g(x)=\dfrac{8-4x}{3}+1=0 \Rightarrow 8-4x=-3 \Rightarrow x=\underline{\mathbf{\dfrac{11}{4}}}$.

B) Given: $y=f(x)=\dfrac{k}{x+2}$ To find the inverse function f^{-1}, we interchange x and y and then resolve for y in terms of x.

$x=\dfrac{k}{y+2} \Rightarrow xy+2x=k \Rightarrow y=f^{-1}(x)=\dfrac{k-2x}{x}$

Thus, $f(2)=f^{-1}(4)\cdot f(4) \Leftrightarrow \dfrac{k}{4}=\dfrac{k-8}{4}\cdot\dfrac{k}{6} \Leftrightarrow \dfrac{1}{4}=\dfrac{k-8}{24} \Rightarrow k=\underline{\mathbf{14}}$.

Solution #2:
Without bothering to explicitly find $y=f^{-1}(x)$, we note that any input to f^{-1} is output from f and solve $4=\dfrac{k}{x+2}$ for x. Cross multiplying, $k=4x+8 \Rightarrow x=\dfrac{k-8}{4}$ and the same result follows.

C) Since the given line with slope 9 passes through (2, 0), its equation must be $y=9x-18$.
At the points of intersection, $f(x)=x^3-6x^2-4x+24=9x-18 \Leftrightarrow x^3-6x^2-13x+42=0$
Since we know that $x=2$ is a root of this equation, we can use synthetic substitution to determine the other roots.

$$\begin{array}{ccccc} & 1 & -6 & -13 & 42 \\ 2 & 1 & -4 & -21 & 0 \end{array}$$

$\Rightarrow x^3-6x^2-4x+42=(x-2)(x^2-4x-21)=(x-2)(x-7)(x+3)=0$

$\Rightarrow x=7,-3$

Substituting in $y=9x-18$, the coordinates of the points of intersection are $\underline{\mathbf{(-3,-45),\ (7,45)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017
ROUND 2 ARITHMETIC / NUMBER THEORY

ANSWERS

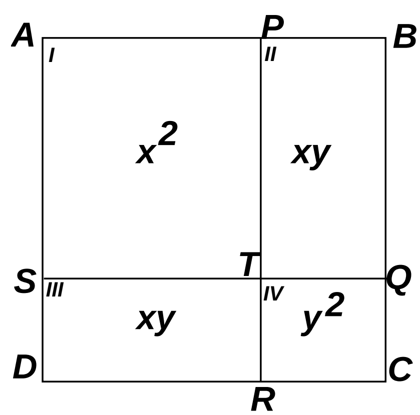
A) ________________________________

B) ________________________________

C) ________________________________

A) Let A and B be nonzero 1-digit or 2-digit natural numbers, where $A+B=13$.
Let $N=\underline{A}\,\underline{B}$ denote either a 2-digit or 3-digit natural number.
Find all possible primes of the form $\underline{A}\,\underline{B}$.

B) Square $ABCD$ has an area of 361.
The 4 rectangles inside $ABCD$ have integer dimensions, but none have areas which are multiples of 3.
Compute all possible areas for rectangle II.



C) Suppose $17.76 is paid out using any combination of pennies, nickels, dimes and/or quarters, with at most 45 of each type of coin. Compute the positive difference between the greatest and least number of coins possible.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017 SOLUTION KEY

Round 2

A) $(1,12) \Rightarrow 112, 121$ (both rejected, $121 = 11^2$)

$(2,11) \Rightarrow 211, \cancel{112}$ $(3,10) \Rightarrow \cancel{310}, 103$ $(4,9) \Rightarrow \cancel{49}, \cancel{94}$

$(5,8) \Rightarrow \cancel{58}, \cancel{85}$ $(6,7) \Rightarrow 67, \cancel{76}$

Additional ordered pairs repeat the same candidates in reverse order.
Thus, the only primes are **<u>67</u>**, **<u>103</u>** and **<u>211</u>** (in any order).
Recall: Testing *N* for primality requires trying prime divisors which when squared are less than *N*.
For *N* = 67, we had to test 2, 3, 5 and 7 only. ($11^2 = 121 > 67$)
For *N* = 103, we had to test these same divisors. $11^2 = 121 > 103$
For *N* = 211, we had to test 2, 3, 5, 7, 11, and 13. ($17^2 = 289 > 211$)

B) Since the side of square *ABCD* is 19, we require pairs $\langle x, y\rangle$ such that $x + y = 19$ and both *x* and *y* are integers, but neither is a multiple of 3. The only possible pairs are $\langle 2,17\rangle$, $\langle 5,14\rangle$ and $\langle 8,11\rangle$ which result in areas of **<u>34</u>**, **<u>70</u>**, and **<u>88</u>**.

C) The ***<u>greatest</u>*** number of coins could be obtained by using the largest number of smaller denominations as possible. Since the number of pennies must be 1 more than a multiple of 5, we use 41 pennies.
$\$17.76 - \$0.41 = \$17.35$
We can use the maximum number of nickels, namely 45.
$\$17.35 - \$2.25 = \$15.10$
The number of dimes must leave a balance that is a multiple of 25.
Thus, 41 dimes leaves $\$15.10 - \$4.10 = \$11.00$ and 44 quarters are required.
The maximum total appears to be $P + N + D + Q = 41 + 45 + 41 + 44 = 171$ coins.
Replacing one quarter with 2 dimes and a nickel, or 5 nickels would increase the total number of coins, but neither of these alternatives would work, since the maximum allowable number of dimes or nickels would be exceeded. **However**, one less quarter, one less nickel and 3 more dimes would maintain the total value and increase the number of coins by 1, without exceeding 45 of any one coin.
The maximum number of coins is $P + N + D + Q = 41 + 44 + 44 + 43 = \underline{172}$.

For the *<u>smallest</u>* number of coins, we must use as many of the larger denominations as possible.
45 quarters $\Rightarrow \$17.76 - \$11.25 = \$6.51$
45 dimes $\Rightarrow \$6.51 - \$4.50 - \$2.01$
40 nickels $\Rightarrow \$2.01 - \$2.00 = \$0.01$
Since the maximum number of quarters is being used, replacing some combination of nickels and dimes with a single quarter is <u>not</u> possible.
Thus, the minimum number of coins is $Q + D + N + P = 45 + 45 + 40 + 1 = \underline{131}$ coins.
The positive difference is **<u>41</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Find a simplified expression for $1+\tan^2 x$ strictly in terms of $\sin x$, where $x \neq \frac{\pi}{2} + k\pi$ for all integers *k*. The expression must be in the form $\frac{N}{D}$, where $N > 0$ for all values of *x*.

B) Solve for θ over $[0, 2\pi)$. $\quad \sin 2\theta = \tan\theta$

C) Solve for *x*: $\quad \operatorname{Arccos}(x) + 2\operatorname{Arcsin}(-1) = -\frac{\pi}{6}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017 SOLUTION KEY

Round 3

A) $1+\tan^2 x \Leftrightarrow 1+\dfrac{\sin^2 x}{\cos^2 x}=\dfrac{\cos^2 x+\sin^2 x}{\cos^2}=\dfrac{1}{\cos^2 x}=\underline{\mathbf{\dfrac{1}{1-\sin^2 x}}}$

B) $\sin 2\theta=\tan\theta \Leftrightarrow 2\sin\theta\cos\theta-\dfrac{\sin\theta}{\cos\theta}=0\Leftrightarrow$

$\dfrac{2\sin\theta\cos^2\theta-\sin\theta}{\cos\theta}=\dfrac{\sin\theta\left(2\cos^2\theta-1\right)}{\cos\theta}=\tan\theta\left(2\cos^2\theta-1\right)=0$

$\Rightarrow \tan\theta=0\Rightarrow\theta=\underline{\mathbf{0,\ \pi}}$

$\Rightarrow \cos\theta=\pm\dfrac{\sqrt{2}}{2}\Rightarrow\theta=\underline{\mathbf{\dfrac{\pi}{4},\dfrac{3\pi}{4},\dfrac{5\pi}{4},\dfrac{7\pi}{4}}}$

C) $\operatorname{Arccos}(x)+2\operatorname{Arcsin}(-1)=-\dfrac{\pi}{6}$

$2\operatorname{Arcsin}(-1)=2\left(-\dfrac{\pi}{2}\right)=-\pi$ (Since, by definition, $\operatorname{Arcsin} A=x\Leftrightarrow\sin x=A$ and $-\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}$)

Solution #1: (Appealing only to the definition of inverse functions)

Substituting, $\operatorname{Arccos}(x)\ +\ 2\operatorname{Arcsin}(-1)\ =-\pi/6\Leftrightarrow\boxed{\operatorname{Arccos}(x)=\dfrac{5\pi}{6}}$.

Since $\operatorname{Arccos} A=x\Leftrightarrow\cos x=A$ and $0\le A\le\pi$,

$\operatorname{Arccos}(x)=\dfrac{5\pi}{6}\Leftrightarrow x=\cos\left(\dfrac{5\pi}{6}\right)\Leftrightarrow x=\underline{\mathbf{-\dfrac{\sqrt{3}}{2}}}$.

Solution #2: (Taking the sine of both sides)
Let θ = Arccos(x). By definition, this is equivalent to $x=\cos\theta$ and $0\le\theta\le\pi$.

Thus, $\sin\left(\operatorname{Arccos}(x)+2\operatorname{Arcsin}(-1)\right)=\sin\left(-\dfrac{\pi}{6}\right)\Leftrightarrow\sin(\theta-\pi)=-\dfrac{1}{2}$.

Since the sine is an <u>odd</u> function, $\sin(\theta-\pi)=-\sin(\pi-\theta)=-\sin(\theta)$

$\Rightarrow\ \sin\theta=\dfrac{1}{2}\ \Rightarrow\ \theta=\dfrac{\pi}{6},\dfrac{5\pi}{6}\ \Rightarrow\ x=\cos\theta=\pm\dfrac{\sqrt{3}}{2}$ (but the positive value is extraneous).

Why is that?

$\operatorname{Arc}\cos\left(-\dfrac{\sqrt{3}}{2}\right)=\dfrac{5\pi}{6}$ and $\dfrac{5\pi}{6}-\pi=-\dfrac{\pi}{6}$ [or $150°-180°=-30°$]

$\operatorname{Arc}\cos\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{\pi}{6}$ and $\dfrac{\pi}{6}-\pi\ne-\dfrac{\pi}{6}$ [or $30°-180°\ne-30°$]

Thus, $x=\underline{\mathbf{-\dfrac{\sqrt{3}}{2}}}$ only.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017
ROUND 4 ALG 1: WORD PROBLEMS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) A tub contains 50 liters of a 5% salt solution. Exactly how many liters of water must be added to change the solution to a 3% solution?

B) I drove a distance of 200 miles at 52 mph.
The next day I drove over the same route at 48 mph.
My average speed for the entire 2-day trip was slightly less than 50 mph.
Compute how much less (in mph).

C) Points A and B are located on the surface of a DVD.
It is known that point A is 3" from the center of the disk.
The DVD is turning at 1024 RPM (revolutions per minute).
In the same timeframe, point B travels 62.5% as far as point A.
In terms of π, point B travels $k\pi$ inches in $\dfrac{1}{10{,}000}$ of a second.
Express k as a simplified rational fraction of the form $\dfrac{N}{D}$.

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 4

A) $0.05(50)+0(x)=0.03(50+x)$

$\Leftrightarrow 250=150+3x \Leftrightarrow x=\underline{\boldsymbol{\dfrac{100}{3}}}$

B) The entire trip was 400 miles.

Rate x Time = Distance $\Rightarrow$ Rate = $\dfrac{\text{Distance}}{\text{Time}}$

$$\Rightarrow \frac{400}{\frac{200}{48}+\frac{200}{52}}=\frac{2}{\frac{1}{48}+\frac{1}{52}}=\frac{2}{\frac{52+48}{48\cdot 52}}=\frac{2\cdot 48\cdot 52}{100}=0.96(52)=49.92$$

Thus, my average was **<u>0.08</u>** mph less than 50 mph.

C) Points A and B are travelling on the circumference of concentric circles.

1 revolution for point A is 6π inches; for B, it is $\dfrac{5}{8}(6\pi)=\dfrac{15}{4}\pi$ inches.

At 1024 RPM, in $\dfrac{1}{10,000}$ of a second, point B makes $\dfrac{1024/60}{10,000}$ of a revolution which is

$$\frac{256}{15\cdot 10^4}\cdot\frac{15\pi}{4}=\frac{2^8}{2^4\cdot 5^4}\cdot\frac{\pi}{2^2}=\frac{2^2}{5^4}\pi=\underline{\boldsymbol{\frac{4\pi}{625}}}\Rightarrow k=\underline{\boldsymbol{\frac{4}{625}}}.$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017
ROUND 5 PLANE GEOMETRY: CIRCLES

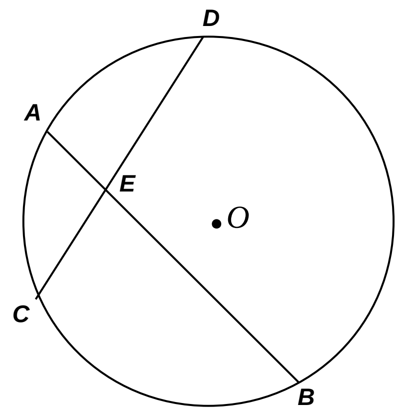
ANSWERS

A) ______________________________

B) ______________________________

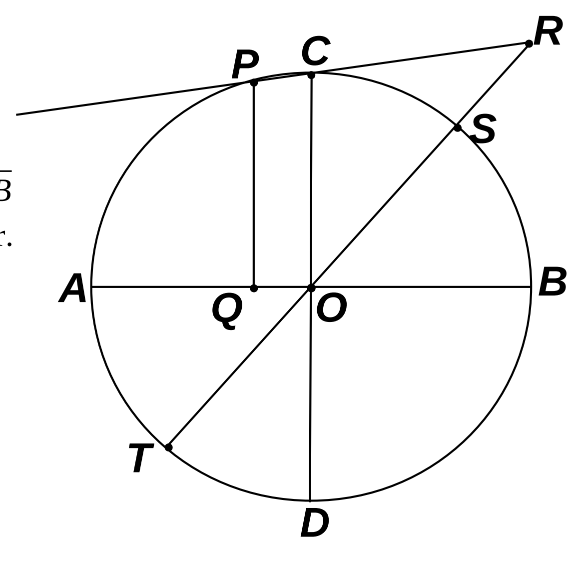
C) ______________________________

A) In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect at point E. If the degree measures of minor arcs $\overset{\frown}{AD}$ and $\overset{\frown}{BC}$ are $(2x)^\circ$ and $(10+3x)^\circ$, respectively, and $m\angle BED = 5(x-1)^\circ$, compute the value of x.



B) Circles are inscribed in and circumscribed about an equilateral triangle with sides of lengths 12. Compute the area of the annulus (the ring between the two circles).

C) $\overline{AB}$ and $\overline{CD}$ are perpendicular diameters. Q is on $\overline{AB}$ such that $\overline{PQ} \parallel \overline{CD}$. Points R, S, O and T are collinear. If $PC = 2$, $CR = 10$ and $RS = 6$, compute PQ.



MASSACHUSETTS MATHEMATICS LEAGUE
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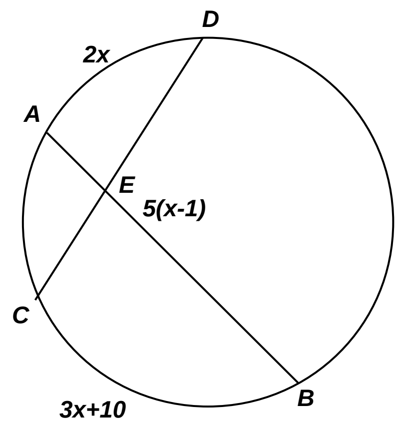
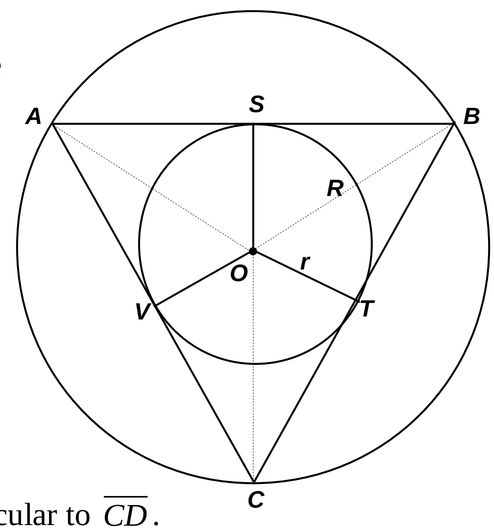
Round 5

A) Since the measure of the vertical angles formed by two intersecting chords is the average of the intercepted arcs, we have



$$\frac{(2x)+(3x+10)}{2}=180-5(x-1)$$
$$\Rightarrow 5x+10=360-10x+10$$
$$\Rightarrow 15x=360 \Rightarrow x=\underline{\mathbf{24}}$$

B) In an equilateral triangle, the medians and altitudes are one and the same segments. Since the medians intersect at a point that divides each median into a 2 : 1 ratio, we see that $r+R=3r$ must equal an altitude of the equilateral triangle, namely, $6\sqrt{3}$. Therefore, $r=2\sqrt{3}$ and $R=4\sqrt{3}$, producing circles with areas of 12π and 48π, and a ring with area $\underline{\mathbf{36\pi}}$.



C) Let $AQ = x$, $PQ = y$ and $TO = OS = r$. Draw $\overline{PE}$ perpendicular to $\overline{CD}$.
Applying the secant-secant rule (outer segment times outer plus inner, i.e. $RC \cdot RP = RS \cdot RT$), we have $10 \cdot 12 = 6(6+2r) \Rightarrow r = 7$.
Applying the product-chord theorem,

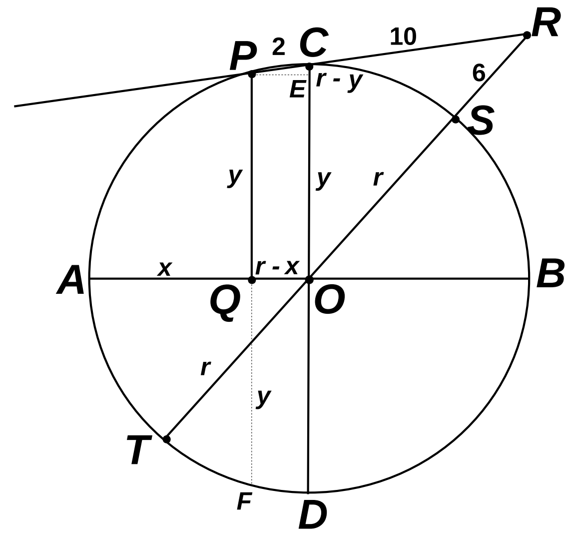
$$AQ \cdot QB = PQ \cdot QF \Rightarrow y^2 = x(14-x) = \boxed{14x-x^2}$$

Applying the Pythagorean Theorem in ΔPEC, we have

$$(7-x)^2+(7-y)^2=2^2$$

Expanding and substituting,

$$\left(49-14x+x^2\right)+\left(49-14y+y^2\right)=4$$
$$94-\left(14x-x^2\right)-14y+y^2=0$$
$$94-14y=0 \Rightarrow y=\underline{\mathbf{\frac{47}{7}}}$$



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) The following list of Pythagorean Triples forms three vertical sequences. The entries in the left column increase by 2. The <u>gaps</u> between successive entries in the middle column are increasing by 4. In each row, the last two entries differ by 1. Compute the sum of the entries in the row beginning with 15.

3	**4**	**5**
5	**12**	**13**
7	**24**	**25**
9	**40**	**41**

B) For some <u>minimum</u> value of *k*, $T=\sum_{n=1}^{n=k} 160(2)^{1-n} > 319$. Compute *k*.

C) The series $3-10+17-24+31-38+\ldots$ contains *n* terms, where *n* is an odd number. The sum of these *n* terms is 87. Compute the n^{th} term.

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 6

A) The next rows are: 11 - 60 - 61, 13 - 84 - 85, 15 - 112 - 113
Thus, the required sum is **240**.

Alternately, $15^2 + x^2 = (x+1)^2 = x^2 + 2x + 1 \Rightarrow 2x = 225 - 1 = 224 \Rightarrow x = 112$ and the same result follows or note that the sum of the entries in the 2nd and 3rd columns are odd perfect squares, namely $4+5=9=3^2$, $12+13=25=5^2$, $24+25=49=7^2$, $40+41=81=9^2, \ldots$. Then the sum that goes with 15 is $15^2 = 225$ and 15 + 225 = **240**.

B) $\sum_{n=1}^{n=k} 160(2)^{1-n} = 160(1) + 160\left(\frac{1}{2}\right) + 160\left(\frac{1}{4}\right) + \ldots = (160+80+40+20+10+5) + (2.5+1.25+\ldots)$
The sum of the first 6 terms is 315.
Keep adding terms until the total exceeds 319.
$= (315) + (2.5 + 1.25 + 0.625 + \ldots)$
$315 \Rightarrow 317.5 \Rightarrow 318.75 \Rightarrow 318.75 + 0.625 > 319 \Rightarrow n = \underline{\mathbf{9}}$.

C) The terms alternate positive (odd)/negative (even).
The sum of two consecutive terms (starting with an odd number) is –7.
Since n is odd, we may assume there are k pairs of terms preceding the n^{th} term.
$k = \frac{n-1}{2}$
Therefore, $-7\left(\frac{n-1}{2}\right) + t_n = 87$.
$t_1 = 3$
$t_3 = 17 = 3 + 14 = 3 + 7 \cdot 2$
$t_5 = 31 = 3 + 28 = 3 + 7 \cdot 4$
...
$t_n = 3 + 7(n-1) = 7n - 4$
$\Rightarrow -7\left(\frac{n-1}{2}\right) + (7n-4) = 87 \Leftrightarrow -7n + 7 + 14n - 8 = 174 \Leftrightarrow 7n = 175 \Rightarrow n = 25$
$\therefore t_{25} = 25 \cdot 7 - 4 = \underline{\mathbf{171}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017
ROUND 7 TEAM QUESTIONS

ANSWERS

A) (__________ , ___________) D) (__________ , ___________)

B) ___________________________ E) (__________ , __________ , __________)

C) ___________________________ F) ___________________________

A) The vertical and horizontal asymptotes of the function $y = \dfrac{1-2x}{x-6}$ intersect at point $P_0(a,b)$.

If this function undergoes the following 6 successive reflections in the given order.

A. across the *x*-axis
B. across the *y*-axis
C. across the horizontal line $y = k$ (where $k < 0$)
D. across the vertical line $x = h$ (where $h > 0$)
E. through the origin and, lastly,
F. across $y = x$,

the new coordinates of P_0 are $(10,-12)$. Compute the ordered pair (h,k) .

B) Anne moves clockwise around a circle and Dick moves counterclockwise. Anne starts at 90° (the 3:00 position) and moves at 13° per second. Dick starts at 210° (the 8:00 position) and moves at 7° per second. They meet at *n* distinct positions on the circle (the first of which is 168°). Compute *n*.
Note: All angles are measured clockwise from 12:00.

C) Compute <u>all</u> values of *x* (in degrees), where $0° \le x < 360°$, that satisfy

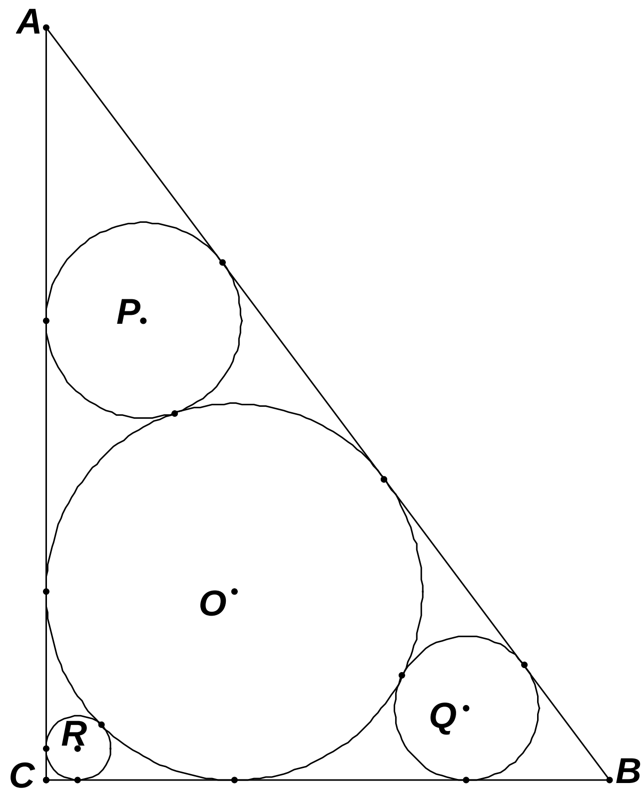
$$\sin 3x + \sin x = \tan\left(Sin^{-1}\frac{2}{\sqrt{5}} \right) \cdot \cos(270° + x)$$

D) A movie theater has a maximum seating capacity of 200. At \$7.50 per ticket, management estimates that 80% of its tickets will be sold for an evening show. Furthermore, for each 25¢ increase in ticket price, ticket sales will decrease by an additional 1% (below the 80% estimate), and, for each 25¢ decrease in ticket price, there will be a 2.5% increase (above the 80% estimate). Management loves to fill all the seats, but even more they like to maximize their profit. Let *T* denote ticket price which maximizes the total revenue *R* earned from the sale of tickets for an evening show. Assuming these assumptions are valid, compute (T,R).

Note: 1% decrease is taken to mean: 80%, 79%, 78%, 77%, 76%, … .
2.5% increase is taken to mean 80%, 82.5%, 85%, 87,5%, 90%., … .

ROUND 7 TEAM QUESTIONS - continued

E) Circle O is inscribed in ΔABC, a 3-4-5 right triangle. Three smaller circles centered at P, Q and R are drawn inside ΔABC.
Each is tangent to circle O and to two sides of ΔABC. Their radii are r_1, r_2 and r_3, where $r_1 < r_2 < r_3$.
Compute the ordered triple (r_1, r_2, r_3).
The diagram <u>is</u> constructed to scale.



F) A sequence, whose initial term is a positive integer t_1, is defined by the following rule:

- if a term is even, the next term is half the current term
- if a term is odd, the next term is 1 more than the current term

For some <u>minimum</u> value of k, for <u>all</u> $n \geq k$, $t_n = 2$ and $t_{n+1} = 1$ (or vice versa).
For example, $t_1 = 48 \Rightarrow 48, 24, 12, 6, 3, 4, 2, 1, 2, 1, 2, 1, \ldots \Rightarrow k = 7$
Compute <u>all</u> values of t_1 for which k = 5.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017 SOLUTION KEY

Team Round

A) Given: $y=\dfrac{1-2x}{x-6}$

Since the denominator cannot be zero, the equation of the vertical asymptote is $x=6$.

Writing $\dfrac{1-2x}{x-6}$ as $\dfrac{\frac{1}{x}-2}{1-\frac{6}{x}}$ we note that as $x\to\pm\infty$ (i.e. gets arbitrarily large), $\frac{1}{x}$ and $\frac{6}{x}$ both approach zero and the quotient approaches –2.

Thus, the equation of the horizontal asymptote is $y = -2$.

Note that:

A reflection of any point through the origin changes the sign of both coordinates of the point.

A reflection of any point across $y = x$ interchanges (swaps) the coordinates of the point.

$P_1(6,-2)$ undergoes the following transformations:

$\Rightarrow A(6,2)\Rightarrow B(-6,2)\Rightarrow C(-6,2k-2)\Rightarrow D(2h+6,2k-2)$

$\Rightarrow E(-2h-6,-2k+2)$

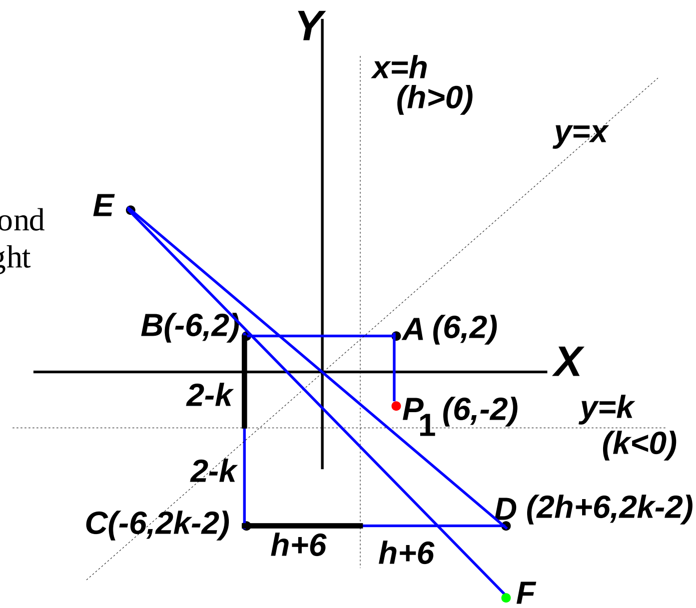
$\Rightarrow F(-2k+2,-2h-6)=(10,-12)$

Therefore, $(h,k)=\underline{(\mathbf{3,-4})}$.

Take care with the order! Equating the first coordinates gives us k, while equating the second coordinates gives us h. The diagram at the right illustrates the sequence of transformations.

Follow the blue line

$P_1(red)\gg A\gg B\gg C\gg D\gg E\gg F(green)$



Check:

$$(6,-2)\underset{x-axis}{\overset{across}{\Rightarrow}}(6,2)\underset{y-axis}{\overset{across}{\Rightarrow}}(-6,2)\underset{\substack{y=-4\\ \Delta y=-6}}{\overset{across}{\Rightarrow}}(-6,-10)\underset{\substack{x=3\\ \Delta x=+9}}{\overset{across}{\Rightarrow}}(12,-10)\underset{origin}{\overset{thru}{\Rightarrow}}(-12,10)\underset{y=x}{\overset{across}{\Rightarrow}}(10,-12)$$

MASSACHUSETTS MATHEMATICS LEAGUE
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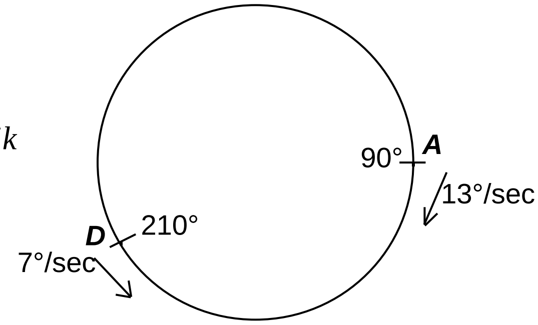
Team Round - continued

B) $90+13k=210-7k \Rightarrow k=6$ [$90+13\cdot 6=210-7\cdot 6=168$ verifies the smallest solution.]

In general, we require that $\begin{cases} 90+13k \equiv n \mod 360 \\ 210-7k \equiv n \mod 360 \end{cases}$



or, $\begin{cases} 90+13k-n=360A \\ 210-7k-n=360B \end{cases} \Leftrightarrow -n=360A-90-13k=360B-210+7k$

$\Leftrightarrow 120-20k=360B-360A$

$\Leftrightarrow k-6=18(A-B)$

$B=A \Rightarrow k=6$

$A=B+1 \Rightarrow k=24$

$A=B+2 \Rightarrow k=42$

The sequence 6, 24, 42, … is generated by $6(3n-2)$ for $n=1,2,3,\ldots$.

Thus, our meeting points are generated by

$90+13(6(3n-2))=90+78(3n-2)=78(3n)-66=\boxed{6(39n-11) \mod 360}$

The sequence generated by this formula is

168, 42, 267, 150, 24, 258, 132, 6, 240, 114, 348, 222, 96, 330, 204, 78, 312, 186, 60, 294, <u>168</u>

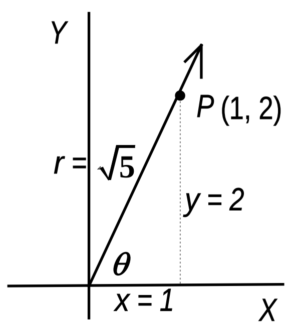
$n=$ <u>**20**</u> distinct values before the cycle repeats.

Is there an easy/easier way to evaluate this sequence or verify the number of distinct solutions without first substituting successive values of n, finding the solution, then subtracting 360s until the result is between 0° and 360° ? Inquiring minds want to know! Send your ideas to <u>olson.re@gmail.com</u>

Is it a coincidence that 13 + 7 = 20?

C) Think of the left hand side as sin(A + B) + sin(A – B) = 2sin$A$cos$B$.

$Sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$ denotes a quadrant 1 value (“angle”) as indicated in the diagram to the right and the tangent of this value is clearly 2.



Using reduction formulas, cos(270° + x) = sin(x). Therefore,

$\sin 3x+\sin x=\tan\left(Sin^{-1}\frac{2}{\sqrt{5}}\right)\cdot\cos(270°+x)$

$\Rightarrow 2\sin 2x\cos x=2\sin x$

$\Rightarrow 4\sin x\cos^2 x-2\sin x=2\sin x\left(2\cos^2 x-1\right)=0$

$\Rightarrow \sin x=0,\ \cos x=\pm\frac{\sqrt{2}}{2}$

$\Rightarrow x=$ <u>**0, 180, 45, 135, 225, 315**</u>

(The degree symbols may be included.)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017 SOLUTION KEY

Team Round - continued

D) Note: \$1 increase in ticket price produces a 4% decrease in sales; \$1 decrease in ticket price produces a 10% increase in ticket sales.

Solution #1 (strictly arithmetic):
@7.50 160 ticket are sold $\Rightarrow$ \$1200
For cheaper ticket prices, we fill more seats, *but make less money*.
[$6.50(180)=1170$, $5.50(200)=1100$]
For more expensive tickets, we fill fewer seats, *but make more money (up to a point!)*.
@8.50 $200(.76)=152$ tickets are sold $\Rightarrow R$ = \$1292
@10.50 $200(.68)=136$ tickets are sold $\Rightarrow R$ = \$1428
@12.50 $200(.60)=120$ tickets are sold $\Rightarrow R$ = \$1500
Increases are slowing down!!
@13.50 $200(.56)=112$ tickets are sold $\Rightarrow R$ = \$1512
@14.50 $200(.52)=104$ tickets are sold $\Rightarrow R$ = \$1508
This *suggests* that T is around \$13.50, but more fine tuning (in 25¢ increments) might produce a larger value of R.
Rather than brute forcing \$13.25 and \$13.75, let's consider an algebraic approach.

Solution #2 (algebraic):
Assuming each \$1 increase produces a 4% decrease in sales.

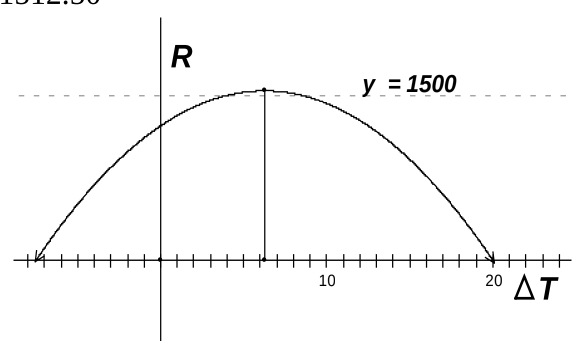
$$R=(7.5+n)(0.8-0.04n)200=(15+2n)(80-4n)=-8n^2+100n+1200$$

$$\Leftrightarrow R=-8\left(n^2-\frac{25}{2}n+\boxed{\left(\frac{25}{4}\right)^2}\right)+1200+8\boxed{\left(\frac{25}{4}\right)^2}=-8\left(n-\frac{25}{4}\right)^2+1512.50$$

$\Leftrightarrow n=\dfrac{25}{4}$ produces the maximum ticket sales of $R=\$1512.50$

$n=\dfrac{25}{4}\Rightarrow T=\$7.50+\$6.25=\$13.75$.

Thus, $(T,R)=\underline{(\mathbf{13.75,\ 1512.50})}$.



Team Round - continued

E) First we find the radius of circle O. Since the radius of any circle inscribed in a triangle is given by the triangle's area divided by its semi-perimeter, we have $r = EO = \dfrac{\frac{1}{2}\cdot 3\cdot 4}{\frac{(3+4+5)}{2}} = \dfrac{6}{6} = 1$.

Concentrate on circle P in the diagram at the left below.

$AE = 3, EO = 1 \Rightarrow AO = \sqrt{10}$

Let $DP = r$ and $AD = x$. Note that $\Delta DAP \sim \Delta EAO$.

$\dfrac{r}{1} = \dfrac{x}{3} = \dfrac{AP}{\sqrt{10}} \Rightarrow x = 3r \Rightarrow AP = r\sqrt{10}$

$AP + PO = AO \Leftrightarrow r\sqrt{10} + r + 1 = \sqrt{10} \Rightarrow r = \dfrac{\sqrt{10}-1}{\sqrt{10}+1}$. Rationalizing, $r_3 = \underline{\boldsymbol{\dfrac{11-2\sqrt{10}}{9}}}$.

Concentrate on circle Q in the middle diagram below.

$BH = 2, OH = 1 \Rightarrow OB = \sqrt{5}$

Let $QG = r$ and $BG = x$. Note that $\Delta GBQ \sim \Delta HBO \Rightarrow \dfrac{r}{1} = \dfrac{x}{2} = \dfrac{BQ}{\sqrt{5}} \Rightarrow x = 2r \Rightarrow BQ = r\sqrt{5}$

$BQ + QO = OB \Leftrightarrow r\sqrt{5} + r + 1 = \sqrt{5} \Rightarrow r = \dfrac{\sqrt{5}-1}{\sqrt{5}+1}$. Rationalizing, $r_2 = \underline{\boldsymbol{\dfrac{3-\sqrt{5}}{2}}}$.

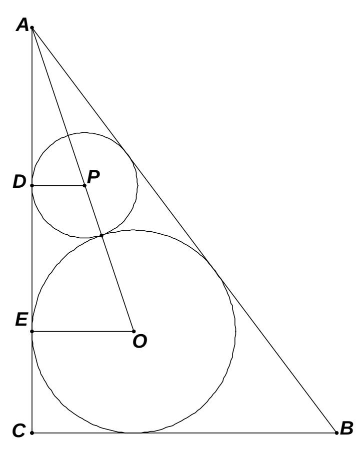
Concentrate on circle R in the diagram at the right below.

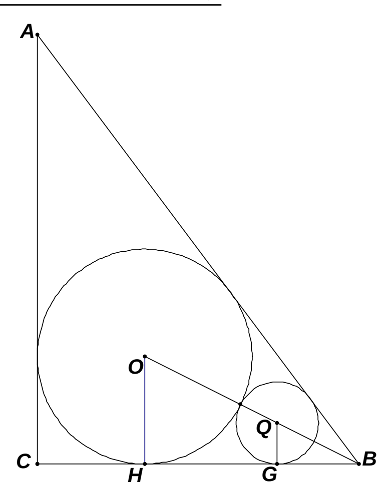
Let $RK = r$ and $CK = x$. Note that $\Delta KCR \sim \Delta JCO$.

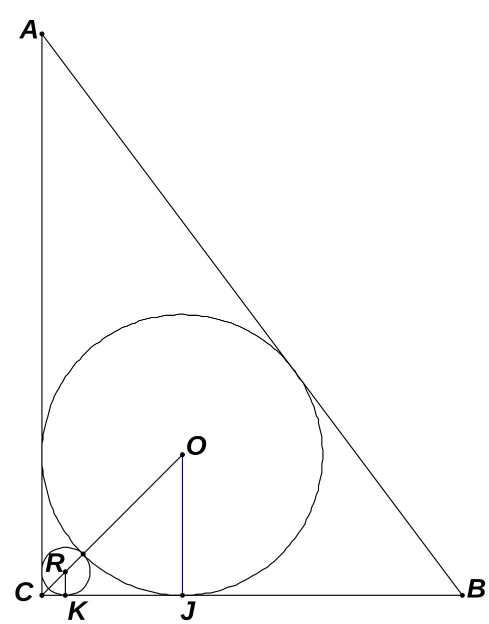
$\dfrac{r}{1} = \dfrac{x}{1} = \dfrac{CR}{\sqrt{2}} \Rightarrow x = r \Rightarrow CR = r\sqrt{2}$. $CR + RO = CO \Leftrightarrow r\sqrt{2} + r + 1 = \sqrt{2} \Rightarrow r = \dfrac{\sqrt{2}-1}{\sqrt{2}+1}$

Rationalizing, $r_1 = 3 - 2\sqrt{2}$

Thus, $(r_1, r_2, r_3) = \underline{\left(\boldsymbol{3-2\sqrt{3}, \dfrac{3-\sqrt{5}}{2}, \dfrac{11-2\sqrt{10}}{9}}\right)}$.







MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017 SOLUTION KEY

Team Round - continued

F)

- if a term is even, the next term is half the current term
- if a term is odd, the next term is 1 more than the current term

Following this algorithm, starting with a power of 2, we eventually produce the alternating sequence 2, 1, 2, 1, … .
Starting with 32, we have $\underline{32}$ - 16 - 8 - 4 - 2 - 1, alternating 2 - 1 thereafter.
The 5th term starts the repetition and k = 5, as required.
Let $t_1 = n$.
$n > 32$ will clearly require longer to settle into the 2-1 repetition.
For $n \le 4$, 3 - 4 - 2 - 1 is the longest sequence and k = 3 is rejected.
Here are the sequences where k = 5:
For $n \le 8$, $\underline{5}$ - 6 - 3 - 4 - 2 - 1 ~~6,3,4,2,1~~ $(k = 4)$ ~~7,8,4,2,1~~ $(k = 4)$ ~~8,4,2,1~~ $(k = 3)$
For $n \le 16$, ~~16,8,4,2,1~~ $(k = 4)$ $\underline{15}$ - 16 - 8 - 4 - 2 - 1, $\underline{14}$ - 7 - 8 - 4 - 2 - 1, $\underline{12}$ - 6 - 3 - 4 - 2 - 1
It is left to you to verify that $n = 9,10,11,13$ fail and that for $17 \le n \le 31$, $k \ge 6$ before the sequence settles into a 2 -1 repetition.
Thus, there are 5 possibilities: **5, 12, 14, 15, 32**

There are several instances of k = 6
For example, start with 28.
Note that if k = 6 for these sequences, starting with 27 or 29 will settle for k = 7.
What patterns do you see?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2017 ANSWERS

Round 1 Algebra 2: Algebraic Functions

A) $\frac{11}{4}$ B) 14 C) $(-3,-45),\ (7,45)$

Round 2 Arithmetic/ Number Theory

A) 67, 103, 211 B) 34, 70 or 88 C) 41

Round 3 Trig Identities and/or Inverse Functions

A) $\frac{1}{1-\sin^2 x}$ B) $0,\pi,\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$ C) $-\frac{\sqrt{3}}{2}$

Round 4 Algebra 1: Word Problems

A) $\frac{100}{3}$ or $33\frac{1}{3}$ B) 0.08 mph (or $\frac{2}{25}$) C) $\frac{4}{625}$

Round 5 Geometry: Circles

A) 24 B) 36π C) $\frac{47}{7}$

Round 6 Algebra 2: Sequences and Series

A) 240 B) 9 C) 171

Team Round

A) $(3,-4)$

B) 20

C) 0, 180, 45, 135, 225, 315

D) $(13.75,\ 1512.50)$

E) $\left(3-2\sqrt{3},\frac{3-\sqrt{5}}{2},\frac{11-2\sqrt{10}}{9}\right)$

F) 5, 12, 14, 15, 32