MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016
ROUND 1 ALG 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) (____________ , ____________)

B) ______________________________

C) ______________________________

A) Given: $\begin{cases} 3x - 4y = 10 \\ \dfrac{x}{2} + ky = c \end{cases}$

If $k = a$ and $c \neq b$, these equations are inconsistent, i.e. there is no solution.
Compute the ordered pair (a,b).

B) $\begin{bmatrix} A+B & B+k \\ C+k+3 & C+D \end{bmatrix}$ is the 2 x 2 identity matrix.
Compute $(A-B)+(C-D)$.

C) If each of 4 numbers is added to the average of the other three,
the sums produced are: 26, 30, 38 and 42. Determine the four numbers.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016 SOLUTION KEY

Round 1

A) Multiplying the second equation by 6, we have $\begin{cases} 3x-4y=10 \\ \dfrac{x}{2}+ky=c \end{cases} \Leftrightarrow \begin{cases} 3x-4y=10 \\ 3x+6ky=6c \end{cases}$

If $6k=-4$, the lines have the same slope and the lines are either parallel or coincident (identical).
If $6c\neq 10$, the equations would not be identical and there would be no solution.
Since inconsistent linear equations have no solution, we have $(k,c)=\underline{\left(-\dfrac{\mathbf{2}}{\mathbf{3}},\dfrac{\mathbf{5}}{\mathbf{3}}\right)}$.

B) Since the 2 x 2 identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have $\begin{bmatrix} A+B & B+k \\ C+k+3 & C+D \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Leftrightarrow$

$$\begin{cases} A+B=C+D=1 \\ B+k=0 \\ C+k+3=0 \end{cases} \Rightarrow \begin{cases} B=-k \\ A=1+k \\ C=-k-3 \\ D=1-C=4+k \end{cases}$$

Thus, $(A-B)+(C-D)=(1+2k)+(-7-2k)=\underline{\mathbf{-6}}$.

C) Let *x*, *y z* and *w* represent the 4 numbers. Then:

$$\begin{cases} x+\dfrac{1}{3}(y+z+w)=26, \\ \text{etc.} \end{cases} \Leftrightarrow \begin{cases} (1)\ \ 3x+y+z+w=78 \\ (2)\ \ x+3y+z+w=90 \\ (3)\ \ x+y+3z+w=114 \\ (4)\ \ x+y+z+3w=126 \end{cases} \Rightarrow \begin{cases} (2)-(1)\ \ -x+y=6 \\ (3)-(1)\ \ -x+z=18 \\ (4)-(1)\ \ -x+w=24 \end{cases} \Rightarrow$$

$$\begin{cases} y=x+6 \\ z=x+18 \\ w=x+24 \end{cases}$$

Substituting in (1), $3x+(x+6)+(x+18)+(x+24)=78 \Rightarrow 6x=30 \Rightarrow x=5$
$\Rightarrow (x, y, z, w) =$ **<u>(5, 11, 23, 29)</u>**. Numbers may be listed in any order.

Alternatively, **add all 4 equations.**
$6(x+y+z+w)=408 \Rightarrow x+y+z+w=68$
Subtracting this equation from equations (1), …, (4) above, we have
$2x=10,\ 2y=22,\ 2z=46,\ 2w=58$ and the same results follow.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016
ROUND 2 ALG1: EXPONENTS AND RADICALS

ANSWERS

A) ______________________________

B) ______________________________

C) S = { ________________________ }

A) Determine all integers for which $x = \left(\dfrac{x-1}{x+3}\right)^{-1}$.

B) Solve for *x*. $\sqrt{4x-2} - \sqrt{2x} = 1$

C) Let S be the set of all values of $x \geq 5$ for which the fraction

$F = \dfrac{(x-8)(x^2-8x+12)+(x-6)(x^2-10x+16)}{x-\sqrt{20x-96}}$ would be zero, undefined, or indeterminant.

Determine all real numbers that belong to set S.

Recall: $\dfrac{0}{N}$, for $N \neq 0$, is zero.

$\dfrac{0}{0}$ is indeterminate, while $\dfrac{N}{0}$, for $N \neq 0$, is undefined.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016 SOLUTION KEY

Round 2

A) $x = \left(\frac{x-1}{x+3}\right)^{-1} \Leftrightarrow x = \frac{x+3}{x-1}$, provided $x \neq -3$.

Cross multiplying, $x^2 - x = x + 3 \Leftrightarrow x^2 - 2x - 3 = (x-3)(x+1) = 0 \Rightarrow x = \underline{\mathbf{3, -1}}$.

B) Squaring both sides, $\sqrt{4x-2} - \sqrt{2x} = 1 \Rightarrow (4x-2) - 2\sqrt{4x-2}\sqrt{2x} + (2x) = 1$

$\Leftrightarrow 6x - 3 = 2\sqrt{4x-2}\sqrt{2x}$

Squaring both sides again, $36x^2 - 36x + 9 = 32x^2 - 16x \Leftrightarrow$

$4x^2 - 20x + 9 = (2x-1)(2x-9) = 0 \Rightarrow x = \frac{1}{2}, \frac{9}{2}$.

Checking is a must when squaring both sides of an equation since extraneous answers may be introduced.

$x = \frac{1}{2} \Rightarrow \sqrt{0} - \sqrt{1} \neq 1$, rejected.

$x = \frac{9}{2} \Rightarrow \sqrt{16} - \sqrt{9} = 1$, check. Thus, $x = \underline{\mathbf{\frac{9}{2}}}$ only.

C) Let $F = \frac{(x-8)(x^2-8x+12)+(x-6)(x^2-10x+16)}{x-\sqrt{20x-96}} = \frac{N}{D}$. $N = 0 \Leftrightarrow$

$(x-8)(x-6)(x-2)+(x-6)(x-2)(x-8) = 2(x-2)(x-6)(x-8) = 0 \Rightarrow x = \cancel{2}, 6, 8$.

(2 is excluded because of the domain restriction.)

$D = 0 \Leftrightarrow x - \sqrt{20x-96} = 0 \Leftrightarrow x = \sqrt{20x-9x} \Rightarrow x^2 = 20x - 96$

$\Rightarrow x^2 - 20x + 96 = (x-8)(x-12) = 0 \Rightarrow x = 8, 12$.

Therefore, the problematic *x*-values are:

$x = \underline{\mathbf{8}}$ (for which the fraction becomes $\frac{0}{0}$, an indeterminant value) and

$x = \underline{\mathbf{12}}$ (for which the fraction becomes $\frac{-52}{0}$ which is undefined). Thus, $S = \underline{\{\mathbf{6,8,12}\}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016
ROUND 3 TRIGONOMETRY: ANYTHING

ANSWERS

A) (____________ , ____________)

B) ______________________________

C) ______________________________

A) Over $0 \le x < 180°$, the equation $2\sin^2 2x + \sin 2x - 1 = 0$ has k solutions that total T°. Compute the ordered pair (k, T).

B) Find the coordinates (x, y) of the points of intersection between the Cartesian equation $y = 3x$ and the polar equation $r = 3\cos\theta$.

Recall the conversion identities: $x = r\cos\theta$ and $y = r\sin\theta$.

C) Compute all possible values of x over $0 \le x < 360°$ for which

$$\cos 140° - \sin 230° + \cos 100° = 2\cos(-300°)(\cos x°).$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016 SOLUTION KEY

Round 3

A) $2\sin^2 2x+\sin 2x-1=0 \Leftrightarrow (2\sin 2x-1)(\sin 2x+1)=0 \Rightarrow \sin 2x=\frac{1}{2},\,-1$

$$2x=\begin{cases}30^\circ\\150^\circ\\270^\circ\end{cases}+n(360^\circ) \Rightarrow x=\begin{cases}15^\circ\\75^\circ\\135^\circ\end{cases}+n(180^\circ).\ n=0 \Rightarrow x=15,75,135 \Rightarrow (k,T)=\underline{\mathbf{(3,225)}}.$$

B) Using the conversion equations $\begin{cases}x=r\cos\theta\\y=r\sin\theta\\x^2+y^2=r^2\end{cases}$, $r=3\cos\theta \Rightarrow \sqrt{x^2+y^2}=\dfrac{3x}{\sqrt{x^2+y^2}}$

Cross multiplying, $x^2+y^2=3x \Rightarrow x^2-3x+9x^2=0 \Rightarrow 10x^2-3x=x(10x-3)=0$

$\Rightarrow x=0,\dfrac{3}{10} \Rightarrow (x,y)=\underline{\mathbf{(0,\,0),\left(\frac{3}{10},\frac{9}{10}\right)}}.$

C) Using reduction formulas and the fact that cosine is an even function,

$\cos 140^\circ-\sin 230^\circ+\cos 100^\circ=2\cos(-300^\circ)(\cos x^\circ)$

$\Leftrightarrow\ -\cos 40^\circ+\sin 50^\circ-\cos 80^\circ=2\cos(300^\circ)(\cos x^\circ)=2\cos 60^\circ\cos x^\circ=\cos x^\circ$

$\Leftrightarrow\ \cos x^\circ=-(\cos 80^\circ+\cos 40^\circ)+\sin 50^\circ$

Using the Sum and Difference Formulas, this simplifies to

$$\cos x^\circ=-\left(2\cos\frac{80^\circ+40^\circ}{2}\cdot\cos\frac{80^\circ-40^\circ}{2}\right)+\sin 50^\circ=-2\cos 60^\circ\cos 20^\circ+\sin 50^\circ=\sin 50-\cos 20^\circ$$

$\Leftrightarrow$

$$\cos 40^\circ-\cos 20^\circ \Leftrightarrow -2\sin\frac{40^\circ+20^\circ}{2}\sin\frac{40^\circ-20^\circ}{2}=-2\sin 30^\circ\sin 10^\circ=-\sin 10^\circ=-\cos 80^\circ.$$

Since $-\cos 80^\circ$ denotes a negative number, we require the related members of the 80° family in quadrants II and III, where the cosine is negative, namely $180^\circ\pm 80^\circ=$ <u>**100°, 260°**</u>.

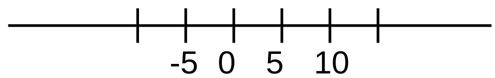
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016
ROUND 4 ALG 1: ANYTHING

ANSWERS

A) _______________________________

B) __________________________ mph

C) (____________ , ____________)



A) A is the average of the largest 3-digit multiple of 5 and the smallest two-digit multiple of 5. If $\dfrac{7x-6}{4}=A$, compute the value of x.

B) It is 4 miles to my school. On a day when I feel particularly ambitious, I start out running, then slow to a walk for a while, and finish the trek jogging.
The distances I run, walk and jog are in a ratio of 8 : 1 : 3 respectively.
The rates at which I run, walk and jog are 12 mph, 3 mph and 6 mph, respectively.
Compute my average rate (in mph) over the 4 miles.

	1:	2:	3:
5:	∧	>	>
4:	∧	♦	∨
3:	∧	∧	∨
2:	∧	∧	∨
1:	♦	<	∨

C) A light display starts in the lower left corner of a grid and spirals <u>clockwise</u> into the center, turning each light on, then off, in sequence. This pattern is illustrated for a 5 x 3 grid in the diagram above. The path follows the directional arrows, moving from the cell in row 1 column 1 to the cell in row 4 column 2.

If the same pattern is executed in a 10 x 10 grid, the 67^{th} light toggled on and off is in row R and column C. Compute the ordered pair (R, C).

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 4

A) $A=\frac{995+(-95)}{2}=450$. Solving for x, $7x-6=1800 \Rightarrow x=\frac{1806}{7}=\underline{\mathbf{258}}$.

B) The distance is irrelevant. Call the run-leg 8*x*, walk-leg *x*, and the jog-leg 3*x*, implying the total distance is 12*x*. Then:

$$D=RT \Leftrightarrow R=\frac{D}{T} \Rightarrow R=\frac{12x}{\frac{8x}{12}+\frac{x}{3}+\frac{3x}{6}}=\frac{12}{\frac{2}{3}+\frac{1}{3}+\frac{1}{2}}=\frac{12}{\frac{3}{2}}=\underline{\mathbf{8}}.$$

C) The path taken will travel up, right, down and left and repeat until all 100 cells are numbered. The chart on the left keeps track of the number of cells in each "leg" of the spiral, the cell number and location of the last cell in each leg. For a relatively small grid, the table on the right is sufficient for numbering the cells in a spiraling pattern. Determining (R,C) of the terminal cell is a matter of reading the row at the left and the column at the top. For a larger grid, where numbering would be *beyond* tedious, the chart on the left hints at the pattern which would lead to a generalization, namely a 4-cycle, with *pairs* of decreasing leg lengths, after the initial leg length which equals the size of the square grid. If the last leg is only partially traversed before the terminal cell is reached, the pattern of doubles will be broken, as is the case in this example**.

Leg Length	Cell #	Row	Column	Direction
10	10	10	1	Right
9	19	10	10	**Down**
9	28	1	10	**Left**
8	36	1	2	**Up**
8	44	9	2	**Right**
7	51	9	9	Down
7	58	2	9	Left
6	64	2	3	Up
3**	67	**<u>5</u>**	**<u>3</u>**	

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
10	<u>10</u>	<u>11</u>	<u>12</u>	<u>13</u>	<u>14</u>	<u>15</u>	<u>16</u>	<u>17</u>	<u>18</u>	<u>19</u>
9	<u>9</u>	<u>44</u>	<u>45</u>	<u>46</u>	<u>47</u>	<u>48</u>	<u>49</u>	<u>50</u>	<u>51</u>	<u>20</u>
8	<u>8</u>	<u>43</u>	<u>70</u>	<u>71</u>	<u>72</u>	<u>73</u>	<u>74</u>	<u>75</u>	<u>52</u>	<u>21</u>
7	<u>7</u>	<u>42</u>	<u>69</u>	<u>88</u>	<u>89</u>	<u>90</u>	<u>91</u>	<u>76</u>	<u>53</u>	<u>22</u>
6	<u>6</u>	<u>41</u>	<u>68</u>	<u>87</u>	<u>98</u>	<u>99</u>	<u>92</u>	<u>77</u>	<u>54</u>	<u>23</u>
5	<u>5</u>	<u>40</u>	<u>67</u>	<u>86</u>	<u>97</u>	<u>100</u>	<u>93</u>	<u>78</u>	<u>55</u>	<u>24</u>
4	<u>4</u>	<u>39</u>	<u>66</u>	<u>85</u>	<u>96</u>	<u>95</u>	<u>94</u>	<u>79</u>	<u>56</u>	<u>25</u>
3	<u>3</u>	<u>38</u>	<u>65</u>	<u>84</u>	<u>83</u>	<u>82</u>	<u>81</u>	<u>80</u>	<u>57</u>	<u>26</u>
2	<u>2</u>	<u>37</u>	<u>64</u>	<u>63</u>	<u>62</u>	<u>61</u>	<u>60</u>	<u>59</u>	<u>58</u>	<u>27</u>
1	<u>1</u>	<u>36</u>	<u>35</u>	<u>34</u>	<u>33</u>	<u>32</u>	<u>31</u>	<u>30</u>	<u>29</u>	<u>28</u>

Thus, $(R,C)=\underline{(\mathbf{5,3})}$.

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 5 PLANE GEOMETRY: ANYTHING

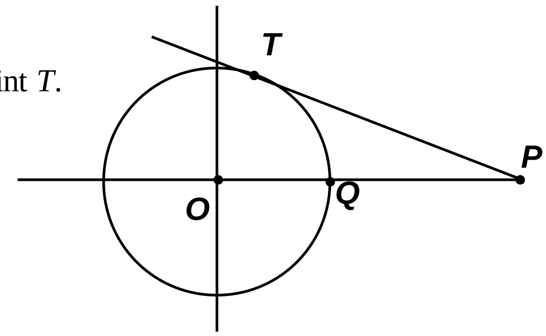
ANSWERS

A) ________________________________

B) ________________________________

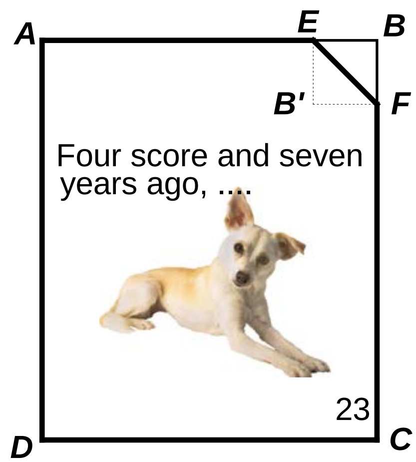
C) ________________________________

A) A tangent line through point P intersects circle O at point T. $PQ = 25$, $PT = 35$. Compute the radius of the circle.

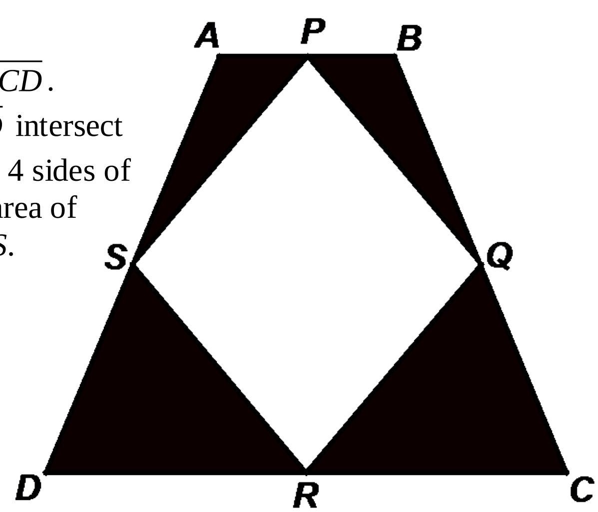


B) Formerly, it was common practice for people were to fold over the corner of the page in a book to hold their place. The page was said to be "dog-eared".
The pentagon $AEFCD$ in the diagram at the right is the dog-eared placeholder of page $ABCD$. I will continue my reading at page 23, when I next pick up the book.

Page $ABCD$ is a rectangle with $AB = 5\frac{1}{4}$ inches and $AD = 6\frac{1}{2}$ inches. If BEF is an isosceles triangle and the area of $AEFCD$ is 33 square inches, compute BE (in inches).

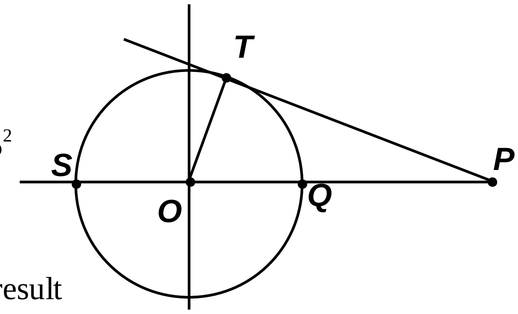


C) $ABCD$ is an isosceles trapezoid with bases $\overline{AB}$ and $\overline{CD}$. $AB:CD = 1:8$. Let the perpendicular from A to $\overline{CD}$ intersect $\overline{CD}$ at point M and $DM = 14$. The midpoints of the 4 sides of the trapezoid (P, Q, R, and S) are connected. If the area of ΔSAP is 24, compute the area of quadrilateral $PQRS$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016 SOLUTION KEY

Round 5



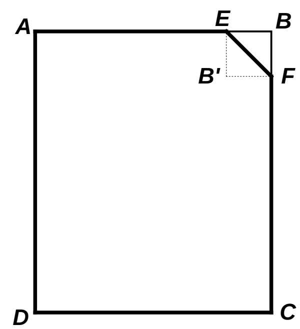
A) Since a radius is perpendicular to any tangent line at the point of tangency, ΔTOP is a right triangle.

Let $OT = OQ = r$. Then: $r^2 + 35^2 = (r+25)^2 \Rightarrow 35^2 = 50r + 25^2$

Taking advantage of the difference of perfect squares,

$50r = 35^2 - 25^2 = (35+25)(35-25) = 60 \cdot 10 = 600 \Rightarrow r = \underline{\mathbf{12}}$.

Alternately, $PQ \cdot PS = PT^2 \Rightarrow 25(25+2r) = 35^2$ and the same result follows.



B) Let $BE = x$. $\mathscr{A}(AEFCD) = \left(5+\frac{1}{4}\right)\left(6+\frac{1}{2}\right) - \frac{1}{2}x^2 = 33$

$\Rightarrow \left(5+\frac{1}{4}\right)\left(6+\frac{1}{2}\right) = 30 + 2.5 + 1.5 + \frac{1}{8} = 34\frac{1}{8} \Rightarrow$

$\frac{1}{2}x^2 = \frac{9}{8} \Rightarrow x^2 = \frac{9}{4} \Rightarrow x = \underline{\mathbf{\frac{3}{2}}}$ or $\underline{\mathbf{1.5}}$.

C) Let $AP = x$. Then:

$DR = 8x$ and $MR = AP \Rightarrow 8x - 14 = x \Rightarrow x = 2$.

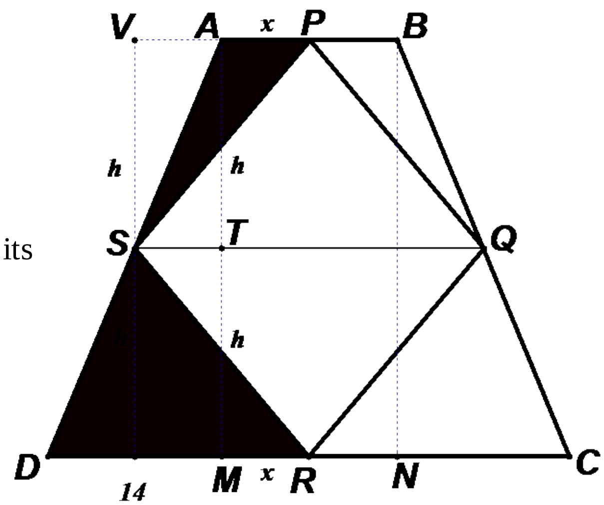
$\begin{cases} AB = 4 \\ CD = 32 \end{cases} \Leftrightarrow QS = 18$

Clearly, $PQRS$ is a rhombus and

$\mathscr{A}(\Delta SAP) = 24 \Rightarrow h = 24 \Rightarrow PR = 48$.

Since the area of a rhombus is half the product of its diagonals, we have the area of $PQRS$ is

$\frac{1}{2} \cdot 18 \cdot 48 = \underline{\mathbf{432}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016
ROUND 6 ALG 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) _____________ : _______________

B) _______________________________

C) _______________________________

A) Two fair 6-sided dice are tossed. Compute the ratio of the probability of the sum showing being a 3, 4 or 5 to the probability of sum showing being a 7, 8 or 9.

B) A binomial expansion with an even exponent will always have an odd numbers of terms.

Determine the value of the middle term of $\left(\frac{1}{x}-x^3\right)^8$.

C) Jar #1 has 4 black and 6 white marbles. Jar #2 has 3 black and 5 white marbles. Two marbles are randomly taken, without replacement, from Jar #1 and put into Jar #2. Then, a marble is randomly taken from Jar #2. Compute the probability that the marble taken from Jar #2 is black.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016 SOLUTION KEY

Round 6

A)

$$\frac{\frac{2+3+4}{36}}{\frac{6+5+4}{36}} = \frac{9}{15} = \underline{\mathbf{\frac{3}{5}}}$$

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

B) The middle term is the 5^{th} term. The 5^{th} term is $\binom{8}{4}(x^{-1})^4\left(x^3\right)^4 = \binom{8}{4}x^8 = \frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}x^8 = \underline{\mathbf{70x^8}}$.

C) Jar #1: 4B and 6W

The two marbles drawn from Jar #1 are either both the same color or one of each.

P(BB) = $\frac{{}_4C_2}{{}_{10}C_2} = \frac{6}{45} = \frac{2}{15}$, P($WW$) = $\frac{{}_6C_2}{{}_{10}C_2} = \frac{15}{45} = \frac{1}{3}$ and P(BW or WB) = $\frac{{}_4C_1 \cdot {}_6C_1}{{}_{10}C_2} = \frac{24}{45} = \frac{8}{15}$

Notice that P(BB) + P(WW) + P(BW or WB) = 1.

Jar #2 initially (3, 5) $\Rightarrow \begin{cases} BB \Rightarrow \text{Jar}\#2:(b,w)=(5,5) \\ WW \Rightarrow \text{Jar}\#2:(b,w)=(3,7) \\ BW/WB \Rightarrow \text{Jar}\#2:(b,w)=(4,6) \end{cases}$

Thus, the probability of drawing B from Jar #2 is

$\frac{2}{15}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{3}{10}+\frac{8}{15}\cdot\frac{2}{5}=\frac{10+15+32}{150}=\underline{\mathbf{\frac{19}{50}}}$ (or **0.38**).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016
ROUND 7 TEAM QUESTIONS
ANSWERS

A) (________ , ________ , ________) D) ________________________ mm^2

B) ____________________________ E) ____________________________

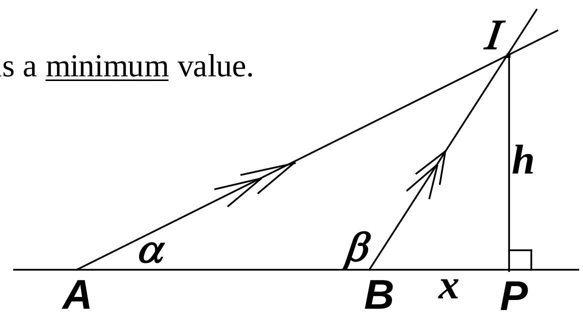
C) (__________ , __________) F) ____________________________

A) Given: $\begin{cases} x+y+z=n \\ 2x-y-z=n+1 \\ x+2y-2z=n+2 \end{cases}$, where *x*, *y*, *z* and *n* are positive integers.

If $x+y>100$, compute (x,y,n) for which $x+y+n$ has a <u>minimum</u> value.

B) Given: $\begin{cases} x=t+t^{-1} \\ y=t-t^{-1} \end{cases}$

Compute all possible real values of *y*, if $x=\frac{1}{2}\sqrt{17}$.

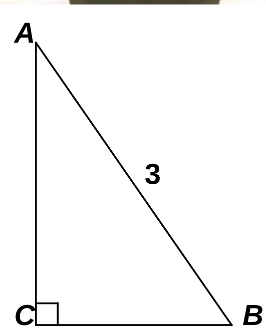


C) A missile fired from point *A* is intercepted at point *I* by a missile fired from point *B*.

If $AB=500, \beta=2\alpha, \tan\alpha=\frac{2\sqrt{3}}{3}$ and β is <u>obtuse</u>, compute the ordered pair (x,h).

D) The Lady Bird Johnson Souvenir Sheet contains 6 stamps.
The dimensions of the sheet are *L* mm by $(L+12)$ mm, where *L* is an integer. Consider each stamp a rectangle whose dimensions (in mm) are integers in an 8 : 5 ratio. The area of the 6 stamps (in mm^2) equals 25% the area of the sheet plus 260 mm^2. Compute the <u>minimum</u> area of the sheet for which this is true.

E) Given: Right ΔABC with hypotenuse *AB* = 3
An arc is drawn with radius *AC* and center *A* intersecting $\overline{AB}$ at point *E*.
An arc is drawn with radius *BC* and center *B* intersecting $\overline{AB}$ at point *D*.
If *DE* = 1.2, compute the area of ΔABC.



F) Solve for *x* over the reals. $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^4=20.25$

MASSACHUSETTS MATHEMATICS LEAGUE
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Team Round

A) $\begin{cases} (1)\ x+y+z=n \\ (2)\ 2x-y-z=n+1 \\ (3)\ x+2y-2z=n+2 \end{cases}$

Subtracting the first two equations, $\boxed{x-2y-2z=1}$.

Subtracting the last two equations, $\boxed{-x+3y-z=1}$.

Adding the boxed equations, $y-3z=2 \Rightarrow y=2+3z$

Substituting in the first boxed equation, $x-4-6z-2z=1 \Rightarrow x=5+8z$

For both of the *z*-expressions for *x* and *y*, as *z* increases, the values of *x* and *y* increases also.

Thus, $x+y=7+11z>100 \Rightarrow z>\frac{93}{11} \Rightarrow z_{\min}=9 \ \Rightarrow (x_{\min}, y_{\min})=(77,29)$

Substituting in equation (1) above, $n=77+29+9=115$.

$\Rightarrow (x,y,n)=\underline{\mathbf{(77,29,115)}}$.

Check: $\begin{cases} 77+29+9=115 \\ 154-29-9=116 \\ 77+58-18=117 \end{cases}$

B) Adding the given equations, we have $x+y=2t \Rightarrow t=\frac{x+y}{2}$

Substituting in the first equation, $x=\frac{x+y}{2}+\frac{2}{x+y} \Rightarrow$

$2x(x+y)=(x+y)^2+4 \Leftrightarrow 2x^2+\cancel{2xy}=x^2+\cancel{2xy}+y^2+4$

$\Rightarrow\ x^2-y^2=4 \Leftrightarrow y=\pm\sqrt{x^2-4}$

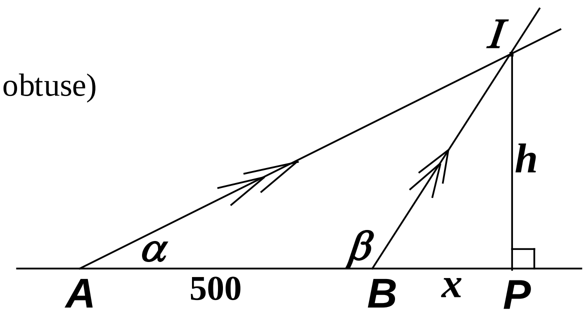
Thus, $y=\pm\sqrt{\frac{17}{4}-4}=\pm\sqrt{\frac{1}{4}}=\underline{\pm\mathbf{\frac{1}{2}}}$.

Alternately, square both equations and subtract.

$\begin{cases} x^2=t^2+2+t^{-2} \\ y^2=t^2-2+t^{-2} \end{cases} \Rightarrow\ x^2-y^2=4$

Substituting, $y^2=\frac{17}{4}-4=\frac{1}{4} \Rightarrow y=\underline{\pm\mathbf{\frac{1}{2}}}$.

Team Round - continued



C) $\tan\alpha = \frac{h}{500+x}$, $\tan\beta = \tan 2\alpha = -\frac{h}{x}$ (since β is obtuse)

$\tan 2\alpha = \frac{2\cancel{\tan\alpha}}{1-\tan^2\alpha} = \frac{-(500+x)\cancel{\tan\alpha}}{x}$.

Cross multiplying, $2x = (500+x)(\tan^2\alpha - 1)$.

Transposing terms,

$2x - x(\tan^2\alpha - 1) = 500(\tan^2\alpha - 1) \Leftrightarrow x(3-\tan^2\alpha) = 500(\tan^2\alpha - 1)$.

$$x = \frac{500(\tan^2\alpha - 1)}{(3-\tan^2\alpha)} = \frac{500\left(\frac{4}{3}-1\right)}{3-\frac{4}{3}} = 500\cdot\frac{1}{3}\cdot\frac{3}{5} = 100.$$

Substituting, $\tan\alpha = \frac{h}{500+x} \Rightarrow \frac{2\sqrt{3}}{3} = \frac{h}{600} \Rightarrow h = 400\sqrt{3} \Rightarrow (x,h) = \underline{\mathbf{(100, 400\sqrt{3})}}$.

D) The area of the 6 stamps is $6\cdot 8x\cdot 5x = 240x^2$ and the area of the sheet is $L(L+12)$.

Therefore,

$$240x^2 = \frac{L(L+12)}{4} + 260 = \frac{L(L+12)+1040}{4} = \frac{(L^2+12L+144)+1040-144}{4} = \frac{(L+12)^2+896}{4}$$

$\Rightarrow (L+12)^2 = 960x^2 - 896 = 64(15x^2 - 14)$

Since x must be an integer, we resort to trial and error until we find the minimum possible value of x for which L is an integer.

$x = 1 \Rightarrow (15x^2 - 14) = 1 \Rightarrow L + 12 = 8$ (rejected).

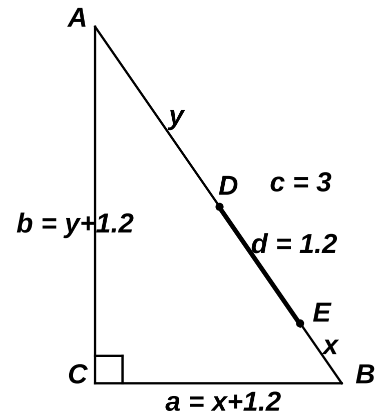
$x = 2 \Rightarrow (15x^2 - 14) = 46$ (rejected).

$x = 3 \Rightarrow (15x^2 - 14) = 135 - 14 = 121 = 11^2 \Rightarrow L + 12 = 8\cdot 11 \Rightarrow L = 76$.

Thus, the minimum dimensions of the sheet are 76 by 88, resulting in an area of **<u>6688</u>** mm^2.

Team Round - continued

E) $\begin{cases} x+y=1.8=\dfrac{9}{5} \\ \left(x+\dfrac{6}{5}\right)^2+\left(y+\dfrac{6}{5}\right)^2=3^2=9 \end{cases}$



$\Rightarrow \left(x+\dfrac{6}{5}\right)^2+\left(\dfrac{9}{5}-x+\dfrac{6}{5}\right)^2=3^2=9 \Rightarrow \left(x+\dfrac{6}{5}\right)^2+(3-x)^2=9$

$\Rightarrow x^2+\dfrac{12}{5}x+\dfrac{36}{25}-6x+x^2=0 \Rightarrow x^2-\dfrac{9}{5}x+\dfrac{18}{25}=0$

$\Rightarrow \left(x-\dfrac{3}{5}\right)\left(x-\dfrac{6}{5}\right)=0 \Rightarrow (x,y)=\left(\dfrac{3}{5},\dfrac{6}{5}\right)$ or $\left(\dfrac{6}{5},\dfrac{3}{5}\right)$

Therefore, the area of ΔABC is $\dfrac{1}{2}(1.8)(2.4)=0.9(2.4)=$ **2.16** .

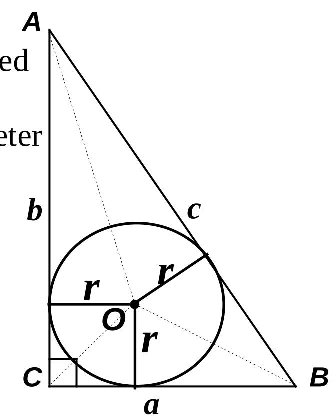
Generalization: Show that DE is equal to the diameter of the inscribed circle in ΔABC.

$DE=d \Rightarrow AD=b-d$, $BE=a-d$, $AB=(b-d)+d+(a-d)=c$. Therefore, $a+b-d=c \Leftrightarrow d=a+b-c$ which is the length of the diameter of the inscribed circle of right triangle *ABC*.

[If r denotes the radius of the inscribed circle in ΔABC and s, the semi-perimeter of ΔABC, then $r\cdot s=\dfrac{1}{2}ar+\dfrac{1}{2}br+\dfrac{1}{2}cr=\dfrac{1}{2}ab \Rightarrow r(a+b+c)=ab$

$\Rightarrow r=\dfrac{ab}{(a+b)+c}$. $(a+b)^2-c^2=2ab$, since $a^2+b^2=c^2$. Factoring,

$(a+b+c)(a+b-c)=2ab \Rightarrow a+b-c=\dfrac{2ab}{a+b+c}=2r=d_{(ic)}$.]



F) $\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^4=20.25 \Leftrightarrow \left(\sqrt{x}\right)^4+4\left(\sqrt{x}\right)^3\left(\dfrac{1}{\sqrt{x}}\right)+6\left(\sqrt{x}\right)^2\left(\dfrac{1}{\sqrt{x}}\right)^2+4\sqrt{x}\left(\dfrac{1}{\sqrt{x}}\right)^3+\left(\dfrac{1}{\sqrt{x}}\right)^4=20.25$

$\Rightarrow x^2+4x+6+\dfrac{4}{x}+\dfrac{1}{x^2}=20\dfrac{1}{4} \Rightarrow \left(x^2+\dfrac{1}{x^2}\right)+4\left(x+\dfrac{1}{x}\right)=20\dfrac{1}{4}-6=14\dfrac{1}{4}=\dfrac{57}{4}$

Let $u=x+\dfrac{1}{x}$. Then: $u^2=x^2+2+\dfrac{1}{x^2}$ or $x^2+\dfrac{1}{x^2}=u^2-2$. Substituting, we have

$(u^2-2)+4u-\dfrac{57}{4}=0 \Leftrightarrow u^2+4u-\dfrac{65}{4}=0 \Leftrightarrow 4u^2+16u-65=(2u-5)(2u+13)=0$

$\Rightarrow u=\dfrac{5}{2}$ only. [$u=-\dfrac{13}{2}=x+\dfrac{1}{x} \Rightarrow 2x^2+13x+2=0$ which has only negative irrational roots.]

Re-substituting, $\left(x+\dfrac{1}{x}=\dfrac{5}{2}\right)\cdot 2x \Rightarrow 2x^2+2-5x=(2x-1)(x-2)=0 \Rightarrow x=$ $\mathbf{2, \dfrac{1}{2}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2016 ANSWERS

Round 1 Algebra 2: Simultaneous Equations and Determinants

A) $\left(-\frac{2}{3},\frac{5}{3}\right)$ B) -6 C) 5, 11, 23, 29 (in any order)

Round 2 Algebra 1: Exponents and Radicals

A) $3, -1$ B) $\frac{9}{2}$ C) $\{6,8,12\}$

Round 3 Trigonometry: Anything

A) $(3,225)$ B) $(0,0), \left(\frac{3}{10},\frac{9}{10}\right)$ C) 100°, 260°

Round 4 Algebra 1: Anything

A) 258 B) 8 C) $(5,3)$

Round 5 Plane Geometry: Anything

A) 12 B) 1.5 C) 432

Round 6 Algebra 2: Probability and the Binomial Theorem

A) $\frac{3}{5}$ (or 3 : 5) B) $70x^8$ C) $\frac{19}{50}$ (or 0.38 or 38%)

Team Round

A) $(77,29,115)$ D) 6688 mm^2

B) $\pm\frac{1}{2}$ E) 2.16

C) $\left(100,400\sqrt{3}\right)$ F) $2, \frac{1}{2}$