MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2016
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

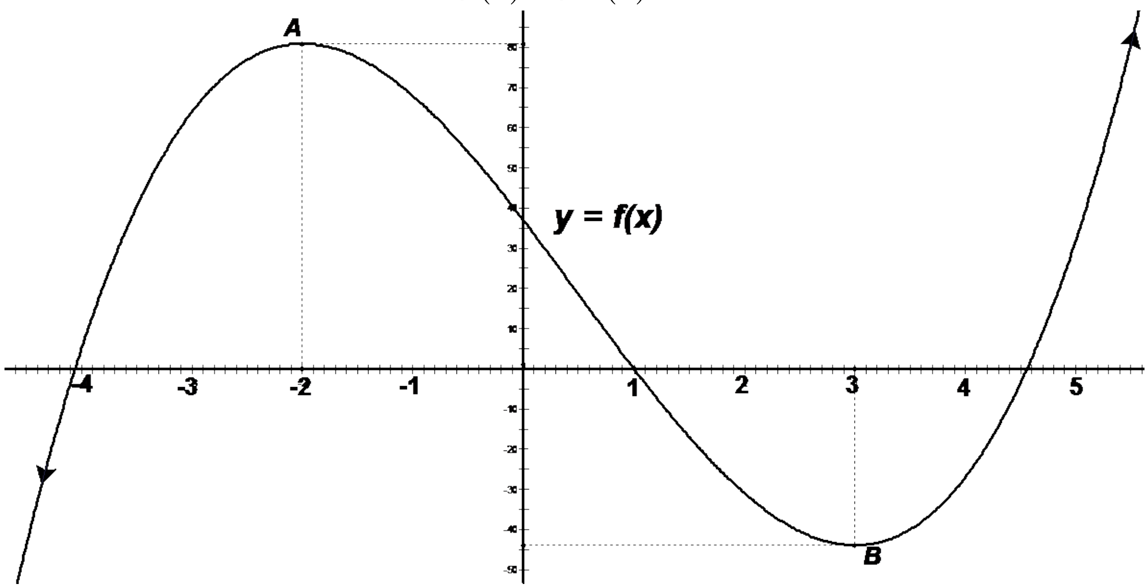
ANSWERS

A) ______________________________

B) 1 2 3 4 5

C) (________ , ________ , ________)

A) Let $f(x) = 11 - 3x$. Compute $\dfrac{1}{f(2)} - \dfrac{1}{f^{-1}(2)}$.



B) Which of these statements about the graph of the cubic polynomial function $y = f(x)$ are true? Circle the correct answer(s) in the answer blank above.

1) The scale used on the *x*-axis is the same as the scale used on the *y*-axis..
2) The maximum value of the function occurs at point *A*.
3) There are <u>exactly</u> three values of *x* for which $f(x) = 0$.
4) If $-2 < a < b < c < 3$, then $f(a) < f(b) < f(c)$.
5) If $f(a) < 0$ and $f(b) > 0$, then for some *c* between *a* and *b*, $f(c) = 0$.

C) The zeros of $f(x) = ax^2 + bx + c$ are r_1 and r_2. If these zeros are each increased by 1, their product triples. Compute the ordered triple (*a*, *b*, *c*), where *a*, *b* and *c* are integers and $a > 0$, for which the original roots are in a 2 : 1 ratio and the sum *a* + *b* + *c* has a <u>maximum</u> value.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2016 SOLUTION KEY

Round 1

A) $f(2) = 11 - 3\cdot 2 = 5$

If $f^{-1}(2) = c$, then $f(c) = 2$. Therefore, without specifically finding $f^{-1}(x)$, we have $f(c) = 11 - 3c = 2 \Rightarrow c = 3$ and $\dfrac{1}{f(2)} - \dfrac{1}{f^{-1}(2)} = \dfrac{1}{5} - \dfrac{1}{3} = \dfrac{3-5}{5\cdot 3} = \underline{\mathbf{-\dfrac{2}{15}}}$.

B) 1) Different scales are used on the *x*- and *y*-axes.

2) As *x* increases $(x \to +\infty)$, $y = f(x)$ is unbounded. At point *A*, there is a local maximum only

4) Over the specified interval, $y = f(x)$ is a decreasing function and $f(c) < f(b) < f(a)$.

Thus, only **<u>3</u>** and **<u>5</u>** are true.

C) Given: $r_2 = 2r_1$

The product of the new zeros is

$$(r_1+1)(r_2+1) = r_1r_2 + (r_1+r_2) + 1 = 3r_1r_2 \Rightarrow 2r_1r_2 - r_2 = r_1 + 1 \Rightarrow \boxed{r_2 = \frac{r_1+1}{2r_1-1}} \; (***)$$

Substituting for r_2 in (***), cross multiplying and transposing terms, we have

$$4r_1^2 - 3r_1 - 1 = (4r_1+1)(r_1-1) = 0 \Rightarrow r_1 = -\frac{1}{4}, 1.$$

Alternately, $r_1r_2 = 2r_1^2$ and $(r_1+1)(r_2+1) = (r_1+1)(2r_1+1) = 3r_1r_2 = 3\left(2r_1^2\right)$

$\Rightarrow 4r_1^2 - 3r_1 - 1 = 0$ and the same result follows.

$$r_1 = -\frac{1}{4} \Rightarrow r_2 = -\frac{1}{2} \Rightarrow f(x) = (4x+1)(2x+1) = 8x^2 + 6x + 1 \Rightarrow a+b+c = 15$$

$$r_1 = 1 \Rightarrow r_2 = 2 \Rightarrow f(x) = (x-1)(x-2) = x^2 - 3x + 2 \Rightarrow a+b+c = 0$$

Thus, (*a*, *b*, *c*) = **<u>(8, 6, 1)</u>**.

FYI - To generate similar problems:
If the new product were *A* times the original (instead of triple) and the roots were in an *N* : 1 ratio (instead of 2 : 1), the equations would be

$$\begin{cases} r_2 = Ar_1 \\ r_2 = \dfrac{r_1+1}{(N-1)r_1 - 1} \end{cases} \Rightarrow A(N-1)r_1^2 - (A+1)r_1 - 1 = 0$$ and adjusting (*N*, *A*) so the discriminant

$(A+1)^2 + 4A(N-1) = (A-1)^2 - 4AN$ is a perfect square will generate a quadratic factorable over the integers and, therefore, a function with integer coefficients.

For example, if $A = 4$, then the discriminant $9 + 16N$ must be a perfect square.

$N = 7, 10, 22, \ldots \Rightarrow 121 = 11^2, 169 = 13^2, 361 = 19^2, \ldots \Rightarrow 24r_1^2 - 5r_1 - 1 = (8r_1+1)(3r_1-1) = 0$,

$36r_1^2 - 5r_1 - 1 = (9r_1+1)(4r_1-1) = 0$, $84r_1^2 - 5r_1 - 1 = (12r_1+1)(7r_1-1) = 0$, … .

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016
ROUND 2 ARITHMETIC / NUMBER THEORY

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) The units digit of a 3-digit integer is prime. Its hundreds digit is twice its tens digit. How many of these 3-digit integers are prime?

B) How many three-digit natural numbers have at least one 9?

C) Determine the smallest value of n for which $\frac{n!(n+1)!}{2016^3}$ is an integer perfect square.

Recall: $n!$ (read n factorial) denotes the product of all natural numbers from 1 to n inclusive, i.e. $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2016 SOLUTION KEY

Round 2

A) The prime digits are 2, 3, 5 and 7.
Clearly, if the units digit is 2 or 5, the 3-digit integer will not be prime.
For a units digit of 3, we have 213, 423, 633, and 843 and all of these are divisible by 3.
For a units digit of 7, we have 217, 427, 637 and 847 and all of these are divisible by 7.
Thus, **none** of these 3-digit integer will be prime. Also accept 0 or zero.

B) Over $100 \le n \le 999$, there are 900 3-digit integers. There are $8 \cdot 9 \cdot 9 = 648$ numbers with no 9s.
Thus, $900 - 648 = \underline{\mathbf{252}}$ with at least one nine.

Solution #2: There is only 1 number with exactly 3 - 9s. In both of the following cases, we must account for the fact that 0 cannot be the leftmost digit.

For exactly two 9s, we have $\binom{8}{1}\binom{2}{2} + \binom{9}{1}\binom{2}{1} = 8 + 18 = 26$

For exactly one 9, we have $\binom{8}{1}\binom{9}{1}\binom{2}{1} + \binom{9}{1}\binom{9}{1} = 144 + 81 = 225$

Thus, the total number is **252**.

Solution #3:

Case 1 - There is one 9:
in the first position: 1 x 9 x 9 = 81
in the second position: 8 x 1 x 9 = 72
in the third position: 8 x 9 x 1 = 72

Case 2 - There are two 9s:
when the non-9 is in the first position: 8
when the non-9 is in the second position: 9
when the non-9 is in the first position: 9

Case 3 – There are three 9s: 1

Total: 225 + 26 + 1 = **252**

C) $\dfrac{(n!)(n+1)!}{2016^3} = \dfrac{(n!)^2}{2016^2} \cdot \dfrac{n+1}{2016}$. Since the first factor is a perfect square, $\dfrac{n+1}{2016} = \dfrac{n+1}{2^5 3^2 7}$ must be a perfect square $\Rightarrow \dfrac{n+1}{2 \cdot 7}$ must be a perfect square and $n = \underline{\mathbf{13}}$ is the smallest integer which does this. However, it still remains to be shown that $\dfrac{(n!)(n+1)!}{2016^2}$ will be an integer for n = 13.

Factoring the denominator, we have $\dfrac{n!(n+1)!}{2016^3} = \dfrac{(n!)^2(n+1)}{2^{15}3^6 7^3}$. For $n = 13$,

$\dfrac{(n!)^2}{2016^2} \cdot \dfrac{n+1}{2016} = \dfrac{(13!)^2(14)}{2^{15}3^6 7^3} = \dfrac{\frac{(13!)^2}{7^2}}{2^{14}3^6}$. Clearly, the numerator is an integer, but we must show that this integer is divisible by 14 factors of 2 and 6 factors of 3.

$\dfrac{13!}{7} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 = 2^{1+2+1+3+1+2} \cdot 3^{1+1+2+1} \cdot 5^{1+1} \cdot 11 \cdot 13 = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 11 \cdot 13$

$\left(\dfrac{13!}{7}\right)^2$ contains 20 factors of 2 and 10 factors of 3. Thus, $\dfrac{(n!)(n+1)!}{2016^2}$ is an integer, as required.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) ______________________________

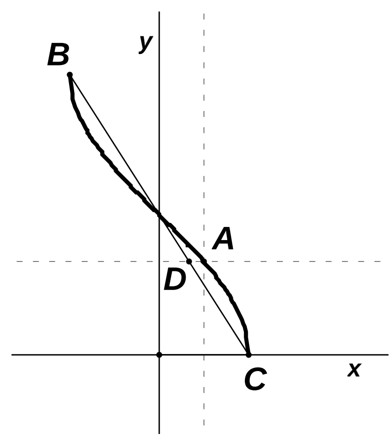
B) ______________________________

C) ______________________________

A) For <u>how many</u> values of A, where $90^\circ < A < 2016^\circ$, is $\sin\theta = \cos(A-\theta)$ an identity?

B) For what values of x over $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ does $\sin(4x) = \cos(2x)$?

C) The vertical line $x = \frac{1}{2}$ intersects $y = Cos^{-1}x$ at point A.
At point B, $y = Cos^{-1}x$ attains its maximum value.
At point C, $y = Cos^{-1}x$ attains its minimum value.
A horizontal line through A intersects line $\overleftrightarrow{BC}$ at point D.
Compute AD.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2016 SOLUTION KEY

Round 3

A) $A = 90 + 360n < 2016 \Rightarrow 450, 810, 1170, 1530, 1890$ - **5** values.

B) $\sin(4x) = \cos(2x) \Leftrightarrow 2\sin(2x)\cos(2x) - \cos(2x) = 0 \Leftrightarrow \cos(2x)(2\sin 2x - 1) = 0$

$\Rightarrow \cos(2x) = 0$ or $\sin(2x) = \dfrac{1}{2}$.

1st condition $\Rightarrow 2x = \dfrac{\pi}{2} + n\pi \Rightarrow x = \dfrac{\pi}{4} + \dfrac{n\pi}{2} = \dfrac{\pi(2n+1)}{4}$ and $n = -1, 0 \Rightarrow \underline{\pm \boldsymbol{\dfrac{\pi}{4}}}$.

2nd condition $\Rightarrow 2x = \dfrac{\pi}{6} + 2n\pi \Rightarrow x = \dfrac{\pi}{12} + n\pi = \dfrac{\pi(12n+1)}{12}$ and $n = 0 \Rightarrow \underline{\boldsymbol{\dfrac{\pi}{12}}}$.

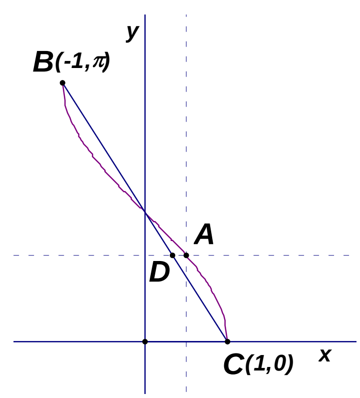
$2x = \dfrac{5\pi}{6} + 2n\pi \Rightarrow x = \dfrac{5\pi}{12} + n\pi = \dfrac{\pi(12n+5)}{12}$ and $n = 0 \Rightarrow \underline{\boldsymbol{\dfrac{5\pi}{12}}}$.

C) $A\left(\dfrac{1}{2}, Cos^{-1}\left(\dfrac{1}{2}\right)\right) = \left(\dfrac{1}{2}, \dfrac{\pi}{3}\right)$, since $\cos\dfrac{\pi}{3} = \dfrac{1}{2}$.

$\overline{AD}$ has equation $y = \dfrac{\pi}{3}$. $\overline{BC}$ has slope $-\dfrac{\pi}{2}$ and

equation $y - 0 = -\dfrac{\pi}{2}(x-1) \Leftrightarrow y = \dfrac{\pi}{2}(1-x)$

$\dfrac{\pi}{3} = \dfrac{\pi}{2}(1-x) \Rightarrow x = \dfrac{1}{3} \Rightarrow AD = \dfrac{1}{2} - \dfrac{1}{3} = \underline{\boldsymbol{\dfrac{1}{6}}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016
ROUND 4 ALG 1: WORD PROBLEMS

ANSWERS

A) ________________________________

B) ________________________________

C) (________ , ________ , ________)

A) In my algebra class, there are 6 rows of 6 seats each.
When all the students are present, no row is empty and no two rows have the same number of students. On a day when two students are absent, how many seats are empty?

B) A soccer player hopes to make 90% of his penalty shots. So far this season, he has gotten two penalty shots per game, and has made 7 of 12 penalty shots. Assuming he continues to get two penalty shots per game and he hits on all of them, how many <u>more</u> games must he play before he reaches 90% accuracy?

C) A group of N people, where $N \geq 100$, were polled for their favorite ice cream.
These people liked 3 different nutty flavors

- A people liked maple walnut (*mw*),
- B people liked pistachio (*p*), and
- C people liked butter pecan (*bp*),

where $N = A + B + C$.
If one <u>more</u> person had liked each flavor, the ratio of responses would have been $mw : p : bp = 3:5:7$. If three <u>fewer</u> people had liked each flavor, the ratio of responses would have been $mw : p : bp = 5:9:13$.
Compute the ordered triple (A, B, C), for a <u>minimal</u> value of N.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016 SOLUTION KEY

Round 4

A) In the 6 rows, there must be, in some order, 1, 2, 3, 4, 5 and 6 students, for a total of 21 students. That leaves 15 empty seats when everyone is present. Thus, there would be **<u>17</u>** empty seats.

B) Let x denote the total number of shots taken over the rest of the season. Then:
$\frac{7+x}{12+x} \geq \frac{9}{10} \Leftrightarrow 70+10x \geq 108+9x \Leftrightarrow x \geq 38 \Rightarrow$ **<u>19</u>** more games.

C) Each 1 more $\Rightarrow 3n+5n+7n = N+3 \Leftrightarrow N = 15n-3$.
Each 3 less $\Rightarrow 5m+9m+13m = N-9 \Leftrightarrow N = 27m+9$.
Equating, $15n-3 = 27m+9 \Rightarrow 5n-9m = 4 \Rightarrow n = \frac{9m+4}{5}$.
$N \geq 100 \Leftrightarrow 15\left(\frac{9m+4}{5}\right)-3 \geq 100 \Leftrightarrow 27m+9 \geq 100 \Rightarrow m \geq 4$.
$m = 4 \Rightarrow n = 8 \Rightarrow (A,B,C) = \underline{\mathbf{(23,39,55)}}$.
Alternate Solution:
$(mw+1):(p+1) = 3:5 \Rightarrow 5(mw+1) = 3(p+1)$.
$(mw-3):(p-3) = 5:9 \Rightarrow 9(mw-3) = 5(p-3)$.
Solving this system of simultaneous equations, we have $(mw,p) = (23,39)$ and, therefore,
$(mw+1):(p+1):(k+1) = 24:40:56$.
Therefore, $N = A+B+C$ and $N \geq 100 \Rightarrow (A,B,C) = \underline{\mathbf{(23,39,55)}}$
for the <u>minimal</u> value of $N = 24(1)+40(1)+56(1)-3 = 117$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016
ROUND 5 PLANE GEOMETRY: CIRCLES

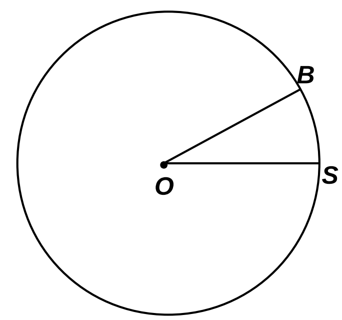
ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

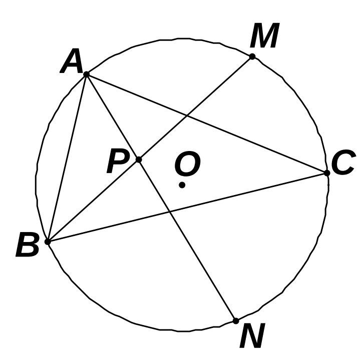
A) In circle *O*, $m\angle BOS = 27°$.
Compute the area of circle *O*, if the area of sector *SOB* is 22.5 .



B) In circle *O*, chord $\overline{AB}$ is a perpendicular bisector of chord $\overline{CD}$.
P is the intersection of $\overline{AB}$ and $\overline{CD}$ and $AP:PB = 1:4$.
If $\overline{PA}, \overline{PC}$, and $\overline{OC}$ have integer lengths, compute the <u>smallest</u> possible area of circle *O*.

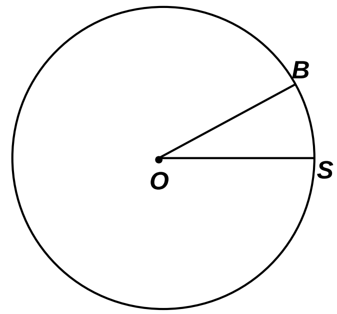
C) Given: ΔABC is inscribed in circle *O* and its interior angle measures (in some order) are in a $5:12:13$ ratio. *M* is the midpoint of minor arc $\overset{\frown}{AC}$, *N* is the midpoint of minor arc $\overset{\frown}{BC}$, and *P* is the intersection of $\overline{AN}$ and $\overline{BM}$. Compute the <u>largest</u> possible degree-measure of <u>obtuse</u> angle *P*.

Note: There are 4 angles with vertex at *P*.



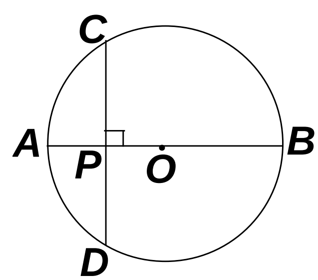
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2016 SOLUTION KEY

Round 5



A) $\frac{27}{360}=\frac{22.5}{A}\Rightarrow\frac{3}{40}=\frac{22.5}{A}\Rightarrow 3A=900\Rightarrow A=\underline{\mathbf{300}}$.

B) Since the perpendicular bisector of any chord passes through the center of the circle, $\overline{AB}$ must be a diameter.



Let $PC=PD=n$ and $PA:PB=m:4m$. Then: $AB=5m$

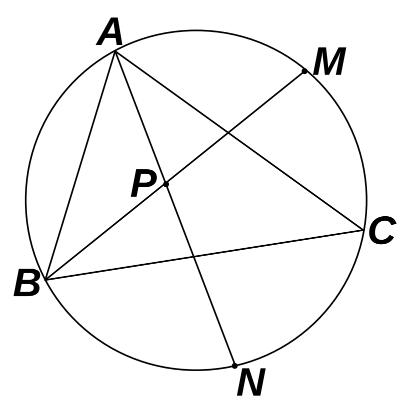
By the product-chord theorem, $n^2=4m^2\Rightarrow n=2m$.

$m=1\Rightarrow n=2$, but radius $OC=2.5$

$m=2\Rightarrow n=4, AB=10, OC=5 \Rightarrow A(\text{circle}\, O)=\underline{\mathbf{25\pi}}$.

C) $5n+12n+13n=180\Rightarrow n=6$

Thus, the measures of the angles of ΔABC are 30°, 72°, and 78°.



1) If $(m\angle A, m\angle B)=(30°,72°), \left(\overset{\frown}{AC},\overset{\frown}{BC}\right)=(60°,144°)$

$\Rightarrow m\angle APM=\frac{30°+72°}{2}=51°$ or $m\angle MPN=180°-51°=129°$

2) If $(m\angle A, m\angle B)=(30°,78°), \left(\overset{\frown}{AC},\overset{\frown}{BC}\right)=(60°,156°)$

$\Rightarrow m\angle APM=\frac{30°+78°}{2}=54°$ or $m\angle MPN=180°-54°=126°$

3) If $(m\angle A, m\angle B)=(72°,78°), \left(\overset{\frown}{AC},\overset{\frown}{BC}\right)=(144°,156°)$

$\Rightarrow m\angle APM=\frac{72°+78°}{2}=75°$ or $m\angle MPN=180°-75°=105°$

Thus, the largest obtuse angle measures $\underline{\mathbf{129}}$°.

An astute observer would have noted that to maximize $m\angle P$, $m\angle A$ and $m\angle B$ had to be minimized and , therefore case 1) above must give the maximum value of *P*.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) ______________________________

B) (________ , ________ , ________)

C) ______________________________

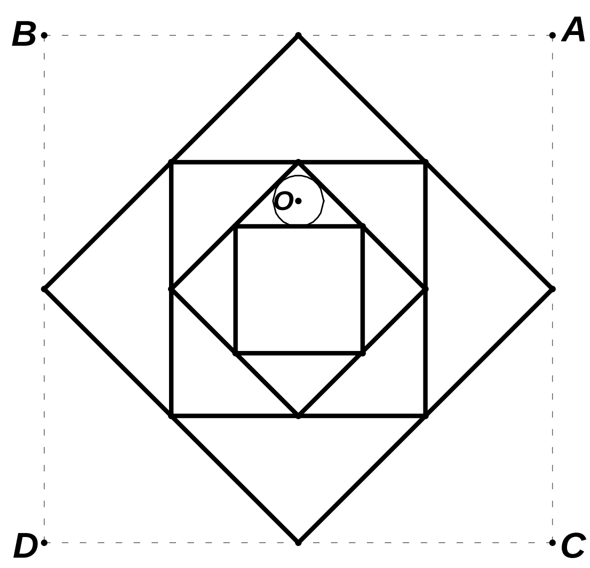
A) Each term in a sequence (except the first) has a value defined in terms of the previous term. Specifically, the rule is $a_n = a_{n-1}\left(a_{n-1}+1\right)$. If $a_1 = 1$, how many terms in this sequence have a value less than 100?

B) There are two different arithmetic progressions of three numbers a_1, a_2, a_3, each of which has the following properties:

- $a_1 + a_2 + a_3 = 30$.
- If 2 is subtracted from the first number, 4 from the second number and 5 from the third number, a geometric progression results.

Compute the ordered triple $\left(a_1, a_2, a_3\right)$, where a_1 is as small as possible.

C) Given: square *ABCD*
A set of nested squares is drawn inside square *ABCD*, where *AB* = 100 cm. The vertices of each nested square are midpoints of the sides of the preceding square, and circle *O* is inscribed in one of the right triangles formed by the 4th and 5th squares as shown. Compute the radius of the circle inscribed in one of the right triangles formed by the 8th and 9th squares.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2016 SOLUTION KEY

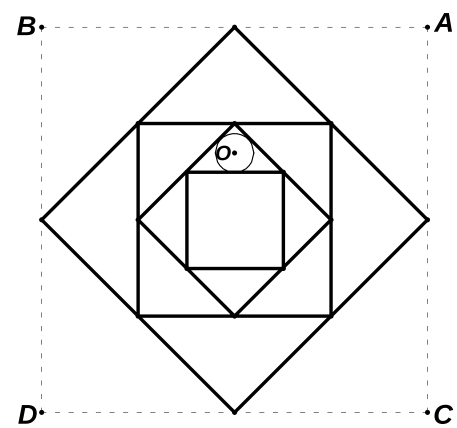
Round 6

A) $a_1 = 1 \Rightarrow a_2 = 1 \cdot 2 = 2 \Rightarrow a_3 = 2 \cdot 3 = 6 \Rightarrow a_4 = 6 \cdot 7 = 42 \Rightarrow a_5 = 42 \cdot 43 > 100$
Thus, only **4** terms are less than 100.

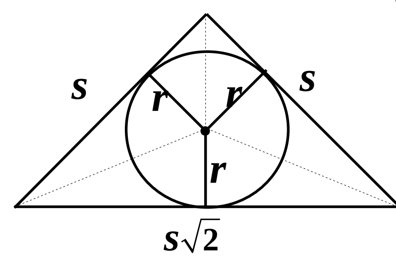
B) If the AP is $x - d$, x, $x + d$, then $3x = 30$ and $x = 10$.
Therefore, the GP is: $8 - d$, 6, $5 + d$ and $\frac{5+d}{6} = \frac{6}{8-d} \Rightarrow 40 + 3d - d^2 = 36$
$d^2 - 3d - 4 = (d-4)(d+1) = 0$
$d = -1 \Rightarrow$ AP: 11, 10, 9
$d = 4 \Rightarrow$ AP: **6, 10, 14**

C) Since the vertices of the each square (after the first) are midpoints of the proceeding square, we have a series of isosceles right triangles whose legs have lengths $50, 25\sqrt{2}, 25, \ldots$, i.e. a geometric progression with $a_1 = 50$ and $r = \frac{\sqrt{2}}{2}$. Thus, the legs of the isosceles right triangle framed by the 8th and 9th squares have length

$$ar^{n-1} = 50\left(\frac{\sqrt{2}}{2}\right)^7 = \frac{50 \cdot 2^3\sqrt{2}}{2^7} = \frac{25\sqrt{2}}{8}$$



To the right is a blowup of the isosceles right triangle and the circle whose radius must be found.



As an isosceles right triangle, its area is given by $\frac{1}{2}s^2$.
As the sum of three isosceles triangles with the same height, the area is given by $2\left(\frac{1}{2}rs\right) + \frac{1}{2}rs\sqrt{2} = \frac{rs}{2}\left(2+\sqrt{2}\right)$.

Equating, we have $s = r\left(2+\sqrt{2}\right) \Rightarrow r = \frac{s}{2+\sqrt{2}} = \frac{s\left(2-\sqrt{2}\right)}{2}$.

Thus, $r = \frac{25\sqrt{2}}{8} \cdot \frac{2-\sqrt{2}}{2} = \frac{25\left(2\sqrt{2}-2\right)}{16} = \boldsymbol{\frac{25\left(\sqrt{2}-1\right)}{8}}$ or equivalent.
(Denominator must be rationalized.)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) (___________ , ___________)

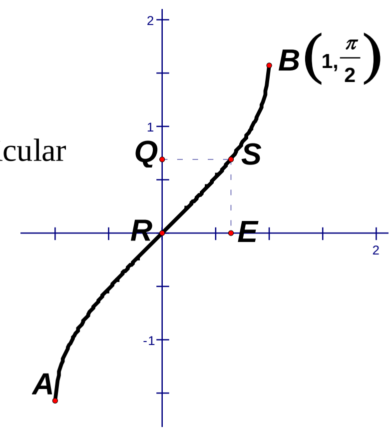
B) C = _____ S = { __________ } E) (_____ , _____ , _____ , _____ , _____)

C) ____________________________ F) ____________________________

A) Given: $f(x)=\dfrac{2x-3}{x+c}$, where $0<c<1$.
Let P be the x-intercept of $y=f(x)$.
Let Q be the y-intercept of $y=f(x)$.
Let R be the intersection of the vertical and horizontal asymptotes of $y=f(x)$.
Compute the unique value of c for which the area of ΔPQR is 6 units^2.

B) Start with any two-digit positive integer N.
Let $C = 0$
Repeat
  Reverse the digits of N to form the integer M.
  Let $T = (N + M) \bmod 100$. [i.e. Add N and M, divide by 100 and save the remainder. Call it T.]
  Let $N = T$ and increase C by 1.
until $T = 99$.
Amazingly, no matter what the starting value is, this is never an infinite loop. Every starting value sooner or later produces 99. For example, $11 \Rightarrow 22 \Rightarrow 44 \Rightarrow 88 \Rightarrow 76 \Rightarrow 43 \Rightarrow 77 \Rightarrow 54 \Rightarrow 99$.
We say that 11 has a cycle of 8, since, for $N=11$, $C=8$. (The loop has been executed 8 times.)
Determine C, the longest cycle and S, the set of all N-values with cycle C.

C) At the right is a graph of the principal inverse sine function.
The domain is $-1 \le x \le 1$ and its range is $-\dfrac{\pi}{2} \le y \le +\dfrac{\pi}{2}$.
Point $S(x,y)=\left(x, Sin^{-1}(x)\right)$ is a point on the graph. $\overline{SQ}$ is perpendicular to the y-axis and $\overline{SE}$ is perpendicular to the x-axis.
$SQRE$ can never actually be a square, that is, the ratio $\dfrac{y}{x}=\dfrac{SE}{SQ}$ can never be exactly 1, but this ratio can be made arbitrarily close to 1 by relocating point S along the curve.

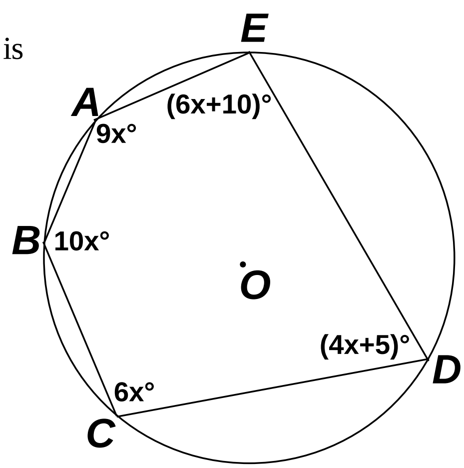


If $Sin^{-1}(x)=\dfrac{3}{5}$, compute $\dfrac{x}{y}$ to the nearest 0.001,
using the identity $\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\ldots+(-1)^{n+1}\dfrac{x^{2n-1}}{(2n-1)!}+\ldots$.
It is necessary to use only the first three terms to approximate the ratio to the nearest 0.001.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016
ROUND 7 TEAM QUESTIONS

D) Two runners start at the same point on a quarter-mile track (440 yards) and run in opposite directions. The speed of the slower runner is R feet/sec.
If the faster runner were to run k feet/sec faster than the slower runner, the runners would pass each other for the first time in 1 minute and 50 seconds.
If the faster runner were to run k times as fast as the slower runner, the runners would pass each other for the first time in 1 minute and 28 seconds.
Compute the ordered pair (f, s), where f denotes the fastest possible speed (in feet/sec) and s denotes the slowest possible speed (in feet/sec) for the faster runner.

E) Pentagon $ABCDE$ (with the indicated angle measures) is inscribed in circle O. Let $(a_1, a_2, a_3, a_4, a_5)$ denote the degree-measures of the 5 arcs subtended by pentagon $ABCDE$, where $a_1 \le a_2 \le a_3 \le a_4 \le a_5$
Compute $(a_1, a_2, a_3, a_4, a_5)$.



F) Let T be the series 2 + 5 + 9 + …, where each term, t_n, denotes the number of diagonals in a polygon with $(n + 3)$ sides. [t_1 = 2 because a quadrilateral (a 4-gon) has 2 diagonals.]
Let S be the sequence of partial sums of T, namely 2, 7, 16, …
Determine the partial sum closest to 2016.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016 SOLUTION KEY

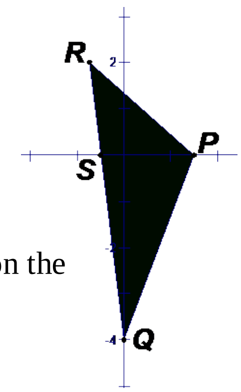
Team Round

A) Since $x=\frac{3}{2}\Rightarrow f(x)=0$, $P\left(\frac{3}{2},0\right)$.

Since $x=0\Rightarrow f\left(x\right)=\frac{-3}{c}$, $Q\left(0,\frac{-3}{c}\right)$.

Since $x=-c$ results in division by 0, $y=f(x)$ has a vertical asymptote at $x=-c$.

Rewriting $f(x)$ as $\dfrac{2-\frac{3}{x}}{1+\frac{c}{x}}$ and letting $x\to\pm\infty$, we see that the horizontal asymptote of f is $y=2$. Thus, $R\left(-c,2\right)$.



Since $0<c<1$, $-\infty<-\frac{3}{c}<-3$, the graph at the right shows Q as some point on the y-axis below $\left(0,-3\right)$ and R as some point between $\left(-1,2\right)$ and $\left(0,2\right)$.

The area of ΔPQR equals the sum of the areas of ΔPSR and ΔPSQ.

Thus, we require that $\frac{1}{2}\cdot PS\cdot 2+\frac{1}{2}\cdot PS\cdot\frac{3}{c}=6\Leftrightarrow PS\left(1+\frac{3}{2c}\right)=6$.

Since the slope of line $\overleftrightarrow{RQ}$ is $\dfrac{2+\frac{3}{c}}{-c}=\dfrac{2c+3}{-c^2}$, the equation of $\overleftrightarrow{RQ}$ is $\left(y-2\right)=\dfrac{2c+3}{-c^2}\left(x+c\right)$.

To find S, we let $y=0$. $x=\dfrac{2c^2}{2c+3}-c=\dfrac{-3c}{2c+3}$ Thus, $S\left(\dfrac{-3c}{2c+3},0\right)$ and $PS=\dfrac{3}{2}+\dfrac{3c}{2c+3}$

Thus, $\left(\dfrac{3}{2}+\dfrac{3c}{2c+3}\right)\left(1+\dfrac{3}{2c}\right)=6\Leftrightarrow\left(\dfrac{12c+9}{2\cancel{(2c+3)}}\right)\left(\dfrac{\cancel{2c+3}}{2c}\right)=6\Rightarrow 12c+9=24c\Leftrightarrow c=\underline{\boldsymbol{\frac{3}{4}}}$.

Team Round - continued

B) Adding N to its reversal frequently produces an integer with identical digits (call them twins).
So we pay attention to the cycles associated with 11, 22, 33, etc.
The number in* [*brackets*] *represents the cycle of the corresponding value of N, i.e. the number of reversals and additions required to reach 99. Note: In the following chains, the cycle numbers were filled in after the chain reached 99.
10 [9] $\Rightarrow$ **11** [8] $\Rightarrow$ **22** [7] $\Rightarrow$ **44** [6] $\Rightarrow$ **88** [5] $\Rightarrow$ 76[4] $\Rightarrow$ 43[3] $\Rightarrow$ **77**[2] $\Rightarrow$ 54[1] $\Rightarrow$ 99
12[14] $\Rightarrow$ **33**[13] $\Rightarrow$ **66**[12] $\Rightarrow$ 32[11] $\Rightarrow$ **55**[10] $\Rightarrow$ 10 continues in above list
13[7] $\Rightarrow$ 44 …. 14[11] $\Rightarrow$ 55 …. 15[13] $\Rightarrow$ 66 … 16[3] $\Rightarrow$ **77** $\Rightarrow$ 54 $\Rightarrow$ 99
This accounts for the cycles associated with all possible twins.

So far 33 has the longest cycle length of any twin, namely 13 and, consequently,
12 has the longest cycle length (14) of any number in the teens.

Larger values of N which eventually produce 33 will have even longer cycles.
Numbers in the 20s, 30s, 40s, etc. produce twins, until they overflow.
We need to concentrate on numbers that produce overflows and eventually 33.
29 $\Rightarrow$ 21 $\Rightarrow$ 33, 38 $\Rightarrow$ 21 $\Rightarrow$ 33, 47 $\Rightarrow$ 21 $\Rightarrow$ 33, 56 $\Rightarrow$ 21 $\Rightarrow$ 33, 65 $\Rightarrow$ 21 $\Rightarrow$ 33 (all cycles of 15).
However, 69 $\Rightarrow$ 65 $\Rightarrow$ 21 $\Rightarrow$ 33 (a cycle of 16).
78 and 87 also have cycle lengths of 16.
But, <u>**89**</u> $\Rightarrow$ 87 $\Rightarrow$ 65 $\Rightarrow$ 21 $\Rightarrow$ 33 and <u>**98**</u> $\Rightarrow$ 87 $\Rightarrow$ 65 $\Rightarrow$ 21 $\Rightarrow$ 33 (both cycles of 17).

C) $y = Sin^{-1}(x) \Rightarrow x = \sin y \;\; \therefore Sin^{-1}(x) = \dfrac{3}{5} \Rightarrow x = \sin\left(\dfrac{3}{5}\right)$.

$$\Rightarrow \frac{x}{y} = \frac{x}{Sin^{-1}(x)} = \frac{\sin\left(\frac{3}{5}\right)}{\frac{3}{5}}.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \Leftrightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots.$$

Plugging in $\dfrac{3}{5}$ (radians) for x, we have $\dfrac{x}{y} = \dfrac{\sin(.6)}{.6} \approx 1 - \dfrac{(.6)^2}{3!} + \dfrac{(.6)^4}{5!} \approx 1 - 0.06 + 0.001 \approx \underline{\mathbf{0.941}}$.

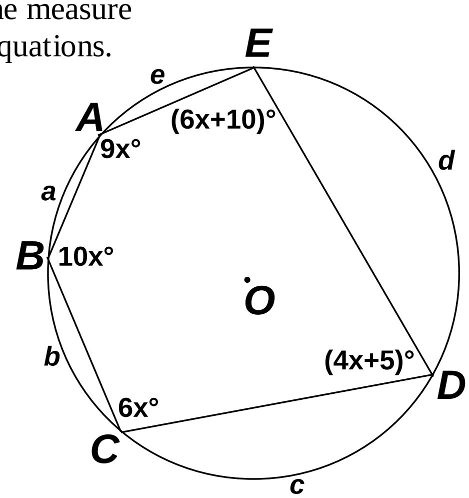
D) $\begin{cases} R\cdot 110 + (R+k)110 = 1320 \\ R\cdot 88 + kR\cdot 88 = 1320 \end{cases} \Rightarrow \begin{cases} 2R + k = 12 \\ R + kR = 15 \end{cases}$

Substituting for k, $R + (12 - 2R)R = 15 \Rightarrow 2R^2 - 13R + 15 = (2R - 3)(R - 5) = 0 \Rightarrow R = 1.5, 5$
$R = 1.5 \Rightarrow k = 9 \Rightarrow$ faster runner runs at $k + R = 10.5$ and $kR = 13.5$ feet/sec.
$R = 5 \Rightarrow k = 2 \Rightarrow$ faster runner runs at $k + R = 7$ and $kR = 10$ feet/sec
Thus, (f, s) = <u>**(13.5, 7)**</u>.

Team Round - continued

E) $9x+10x+6x+(4x+5)+(6x+10)=35x+15=540 \Rightarrow x=\dfrac{525}{35}=15$

Therefore, the angles of $ABCDE$ measure $135°, 150°, 90°, 65°, 100°$.
Since the degree-measure of an intercepted arc is twice the measure of the inscribed angle, we have the following system of equations.



$$\begin{cases} b+c+d=270 \\ c+d+e=300 \\ a+d+e=180 \\ a+b+e=130 \\ a+b+c=200 \end{cases}$$

Subtracting successive equations,

$$\begin{cases} e-b=30 \\ c-a=120 \\ d-b=50 \\ c-e=70 \end{cases}$$

Expressing each arc in terms of the same variable (e),

$$\begin{cases} a=c-120=e-50 \\ b=e-30 \\ c=e+70 \\ d=b+50=e+20 \end{cases}$$

$a+b+c+d+e=5e+10=360 \Rightarrow e=70 \Rightarrow (a,b,c,d)=(20,40,140,90)$

$(a_1,a_2,a_3,a_4,a_5)=\underline{\mathbf{(20,40,70,90,140)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2016 SOLUTION KEY

Team Round - continued

F) The series is 2 + 5 + 9 + 14 + 20 + 27 + 35 + …
The sequence of partial sums is 2, 7, 16, 30, 50, 77, …
The sequence of first differences is: 5, 9, 14, 20, 27, …..
The sequence of second differences is: 4, 5, 6, 7, …..
The sequence of third differences is: 1, 1, 1, 1, …..
Thus, the general term for the sequence of partial sums is a 3rd degree polynomial of the form An^3+Bn^2+Cn+D.
[A system of *difference equations* looks intimidating, but, although tedious, it is actually very easy to solve, in stark contract to a system of *differential equations*. But that's a story for another day.]

Letting n = 1, 2, 3 and 4

Subtracting....

$$\begin{cases} s_1 = A+B+C+D=2 \\ s_2=8A+4B+2C+D=7 \\ s_3=27A+9B+3C+D=16 \\ s_4=64A+16B+4C+D=30 \end{cases} \Rightarrow \begin{cases} 37A+7B+C=14 \\ 19A+5B+C=9 \\ 7A+3B+C=5 \end{cases} \Rightarrow \begin{cases} 18A+2B=5 \\ 12A+2B=4 \end{cases} \Rightarrow A=\frac{1}{6}$$

Substituting back, $(B, C, D) = \left(1,\frac{5}{6},0\right)$. Therefore,

$s_n=\frac{1}{6}n^3+n^2+\frac{5}{6}n=\frac{n^2+6n+5}{6}=\frac{n(n+1)(n+5)}{6}$. We require $\frac{n(n+1)(n+5)}{6}=2016$ or as close as possible. Since the product of the three terms in the numerator is approximately n^3 and $20^3 < 12{,}096 < 30^3$, we start with $n = 20$.

$\frac{20(21)(25)}{6}=10(7)(25)=1750$, $\frac{21(22)(26)}{6}=7(11)(26)=2002$, $\frac{22(23)(27)}{6}=11(23)(9)=2277$

Thus, the closest value is **<u>2002</u>**.

Alternately, let $S(n)=2+7+16+...$ and $A(n)=1+2+3+...+n=n(n+1)/2$.

n	$S(n)$	$A(n)$	$S(n)/A(n)$
1	2	1	2/1 = 6/3
2	7	3	7/3
3	16	6	16/6 = 8/3
4	30	10	30/10 = 9/3
5	50	15	50/15 = 10/3

From this table, we see that $\frac{S(n)}{A(n)}=\frac{n+5}{3}$, so $S(n)=A(n)\cdot\frac{(n+5)}{3}=\frac{n(n+1)(n+5)}{6}$ and have the same result. [Thanks to Norm Swanson - A very slick solution indeed!]

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2016 ANSWERS

Round 1 Algebra 2: Algebraic Functions

A) $-\frac{2}{15}$ B) 3 and 5 C) (8, 6, 1)

Round 2 Arithmetic/ Number Theory

A) none B) 252 C) 13

Round 3 Trig Identities and/or Inverse Functions

A) 5 B) $\pm\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$ C) $\frac{1}{6}$

Round 4 Algebra 1: Word Problems

A) 17 B) 19 C) $(23, 39, 55)$

Round 5 Geometry: Circles

A) 300 B) 25π C) 129

Round 6 Algebra 2: Sequences and Series

A) 4 B) 6 10 14 C) $\frac{25\left(\sqrt{2}-1\right)}{8}$ (or equivalent)

Team Round

A) $\frac{3}{4}$ D) (13.5, 7)

B) $C = 17$ $S = \{89, 98\}$ E) $(20, 40, 70, 90, 140)$

C) 0.941 F) 2002