MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) k = __________________________

A) An equation of the axis of symmetry of the parabola $y=(x+3)(x-h)$ is $x=8$. Compute the value of h.

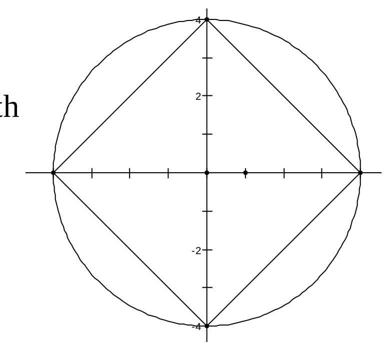
B) The area of the region between $x^2+y^2=16$ and $|x|+|y|=4$ can be expressed as $A(\pi-B)$, where $B>0$. Determine the ordered pair of <u>integers</u> (A, B).

C) The line ℓ defined by the equation $3x + 4y = 24$ passes through one of the endpoints of the major axis and one of the endpoints of minor axis of the ellipse defined by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b$. Compute the y-intercept k of the line perpendicular to ℓ and passing through the focus of the ellipse that lies on the positive x-axis.

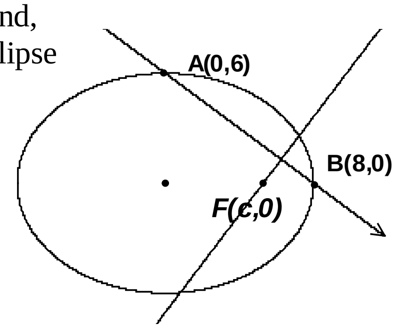
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 SOLUTION KEY

Round 1

A) The x-intercepts of the parabola $y=(x+3)(x-h)$ at $A(-3,0)$ and $B(h,0)$ are images of each other across the axis of symmetry $x=8$. Thus, $(8,0)$ is the midpoint of $\overline{AB}$
$\Rightarrow \dfrac{-3+h}{2}=8 \Rightarrow h=2\cdot 8+3=\underline{\mathbf{19}}$.



B) The given equations determine a circle of radius 4 and a square with diagonals of length 8. Thus, the area of the enclosed region is
$16\pi-\dfrac{1}{2}\cdot 8\cdot 8=16(\pi-2) \Rightarrow (A,B)=\underline{\mathbf{(16,2)}}$.



C) The line $3x + 4y = 24$ passes through points $A(0, 6)$ and $B(8, 0)$ and, since $a>b$, the ellipse is horizontal. Thus, the equation of the ellipse is $\dfrac{x^2}{64}+\dfrac{y^2}{36}=1$. The focus is at $(c, 0)$ and since, for an ellipse, $a^2=b^2+c^2$, we have $c = \sqrt{64-36}=\sqrt{28}=2\sqrt{7}$.
The equation of the perpendicular line must be of the form $4x - 3y = n$, for some constant n. Since the focus is at $\left(2\sqrt{7},0\right)$, we have $4x - 3y = 8\sqrt{7}$. Therefore, $k = \underline{-\dfrac{\mathbf{8\sqrt{7}}}{\mathbf{3}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Solve for x over the rational numbers. $2x = 3 + \frac{15}{x+2}$

B) The difference between the squares of two positive integers A and B is 72.
Compute <u>all</u> possible ordered pairs (A, B), where $A > B$.

C) Factor completely over the integers using the fewest possible number of minus signs.

$$\left(x^2 + y\right)\left(x^2 - y\right) + x^2\left(y^2 - 1\right)$$

Consider expressions like $(a - 1)$ and $\left({}^{-}1 + a\right)$ to each have one minus sign.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 SOLUTION KEY

Round 2

A) $2x = 3 + \dfrac{15}{x+2} \Rightarrow (2x-3)(x+2) = 15 \Leftrightarrow 2x^2 + x - 6 - 15 = 0$, provided $x \neq -2$.

$\Leftrightarrow 2x^2 + x - 21 = (x-3)(2x+7) = 0$

$\Rightarrow x = \underline{\mathbf{3, -\frac{7}{2}}}$.

B) $A^2 - B^2 = 72 \Rightarrow (A+B)(A-B) = 72 \Rightarrow \begin{cases} A+B \\ A-B \end{cases} = \begin{cases} 72 \ \mathbf{36} \ 24 \ \mathbf{18} \ \mathbf{12} \ 9 \\ 1 \quad \mathbf{2} \quad 3 \quad \mathbf{4} \quad \mathbf{6} \ \ 8 \end{cases}$

For <u>positive</u> <u>integers</u> A and B, it is always true that $A + B > A - B$, so we stop at $(A+B, A-B) = (9,8)$. If $A+B$ and $A-B$ were to have opposite parity, then neither A nor B would be integers. Thus, we consider only the bolded pairs.

Adding the equations, we have $2A = \begin{cases} 38 \\ 22 \\ 18 \end{cases} \Rightarrow A = 19, 11, 9 \Rightarrow (A,B) = \underline{\mathbf{(19,17), (11,7), (9,3)}}$

C) $(x^2+y)(x^2-y) + x^2(y^2-1)$ is a sum of two terms and, since there is no common factor between these two terms (other than 1), we have no option other than first multiplying out these terms $(x^2+y)(x^2-y) + x^2(y^2-1) = x^4 - y^2 + x^2y^2 - x^2$.

Grouping the first and third, and second and fourth terms, we have

$(x^4 + x^2y^2) - (y^2 + x^2) = x^2(x^2+y^2) - 1(y^2+x^2) = (x^2+y^2)(x^2-1) = \underline{\mathbf{(x^2+y^2)(x-1)(x+1)}}$

or $\underline{\mathbf{(x^2+y^2)({}^-1+x)(1+x)}}$. Each has exactly one minus sign.

Do <u>not</u> accept

$(x^2+y^2)(1-x)(-1-x)$, $(x^2+y^2)(1+{}^-x)({}^-1-x)$ or $(x^2+y^2)(1+{}^-x)({}^-1+{}^-x)$,

since each of these is considered to have three "minus" signs.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016
ROUND 3 TRIG: EQUATIONS WITH A REASONABLE NUMBER OF SOLUTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Compute <u>all</u> values of x over the interval $\left[0, 2\pi\right]$ for which $3\sin^2 x + 2\cos x + 2 = 0$.

B) The equation $\cos x(2\sin x + 1) = 0$ has exactly 4 solutions over the interval $0° < x \leq k°$
Compute the <u>minimum</u> value of k.

C) Determine the <u>smallest</u> positive degree-measure of θ for which

$$\sin\left(3\theta - 160°\right) = \cos\left(150° - 2\theta\right) .$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 SOLUTION KEY

Round 3

A) $3\sin^2 x + 2\cos x + 2 = 0$

$\Leftrightarrow 3\left(1-\cos^2 x\right) + 2\cos x + 2 = 0$

$\Leftrightarrow 3\cos^2 x - 2\cos x - 5 = \left(3\cos x - 5\right)\left(\cos x + 1\right) = 0$

$\Rightarrow \cos x = \cancel{\frac{5}{3}}, -1 \Rightarrow x = \underline{\boldsymbol{\pi}}$.

B) $\cos x = 0 \Rightarrow x = 90° + 180n \Rightarrow 90°, 270°, \ldots$

$2\sin x + 1 = 0 \Rightarrow \sin x = -\frac{1}{2} \Rightarrow x = \begin{cases} 210° + 360n \\ 330° + 360n \end{cases} \Rightarrow 210°, 570°, \ldots$ or $330°, 690°, \ldots$

Arranging in order of increasing magnitude, 90°, 210°, 270°, 330°, 450°, 570°, 690°, … , we see the minimum value of k is **<u>330</u>**.

C) Converting to sine functions only,

$\sin\left(3\theta - 160°\right) = \cos\left(150° - 2\theta\right) \Leftrightarrow \sin\left(3\theta - 160°\right) = \sin\left(90° - \left(150° - 2\theta\right)\right) = \sin\left(2\theta - 60°\right)$

Noting that $\sin A = \sin B \Rightarrow A = B + n\left(360°\right)$ or $A = 180° - B + n\left(360°\right)$, we examine 2 cases.

Case 1: $3\theta - 160° = 2\theta - 60° + n\left(360°\right) \Rightarrow \theta = 100° + \cancel{n\left(360°\right)}$

(We ignore the co-terminal factor, since we want the <u>smallest</u> positive value.)

Case 2: $3\theta - 160° = 180° - \left(2\theta - 60°\right) + n\left(360°\right)$

$\Rightarrow 5\theta = \left(160 + 180 + 60\right)° + n\left(360°\right) = 400° + n\left(360°\right)$

$\Rightarrow \theta = 80° + 72n$. [80° is NOT the smallest.]

$n = -1$ gives us the <u>smallest</u> positive solution $\theta = \underline{\mathbf{8}}°$.

A Check: For $\theta = 8$,

$\sin\left(3\theta - 160\right) = \sin\left(-136°\right) = -\sin\left(136°\right) = -\sin\left(180° - 44°\right) = -\sin 44°$.

$\cos(150° - 2\theta) = \cos\left(134°\right) = \cos\left(180° - 46°\right) = -\cos\left(46°\right) = -\sin\left(90° - 46°\right) = -\sin 44°$.

You should be able to give a reason for each of the above equalities. For example, the sine is an odd function, or, the cosine of an angle equals the sine of its complement, etc.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) It is not uncommon for an equation, like $x^2 = 38x - N$, to have two real solutions. For example, when N = 280, x = 10 or 28. Determine the <u>unique</u> value of the <u>constant</u> N for which the quadratic equation has <u>exactly one</u> solution.

B) Compute the <u>product</u> of the cubes of the rational roots of $\left(x+\frac{4}{x}\right)^4 - 17\left(x+\frac{4}{x}\right)^2 + 16 = 0$.

C) The equation $x^2 - kx + 2k = 2x$ has one root which is 3 greater than the other root. If the roots are R_1 and R_2, find <u>all</u> ordered triples (R_1, R_2, k), where $R_1 > R_2$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 SOLUTION KEY

Round 4

A) $x^2 = 38x - N \Leftrightarrow x^2 - 38x + \square = 0$.

To have equal roots, this trinomial must be a perfect square (like $(x+3)^2 = x^2 + 6x + 9$).

Note that to have equal roots, the value in the box must be the square of half the coefficient of the middle (i.e. linear) term.

$\Rightarrow N = 361$.

Check: $(x - 19)^2 = x^2 - 38x + \underline{\mathbf{361}} = 0$ has exactly one root, namely 19.

B) Let $y = x + \dfrac{4}{x}$. Then the equation becomes

$y^4 - 17y^2 + 16 = 0 \Rightarrow (y^2 - 1)(y^2 - 16) \Rightarrow y = \pm 1, \pm 4$.

$x + \dfrac{4}{x} = \pm 1 \Leftrightarrow x^2 \pm x + 4 = 0$ which has no rational solutions.

$x + \dfrac{4}{x} = \pm 4 \Leftrightarrow x^2 \pm 4x + 4 = (x \pm 2)^2 = 0 \Rightarrow x = \pm 2$.

$(-2)^3 \cdot (2)^3 = \underline{\mathbf{-64}}$.

C) $x^2 - kx + 2k = 2x \Leftrightarrow x^2 - (k+2)x + 2k = 0$

Let $R_1 = A$ and $R_2 = A - 3$. Then: $\begin{cases} 2A - 3 = k + 2 \\ A(A-3) = 2k \end{cases}$

Substituting $k = 2A - 5$ in the second equation,

$A^2 - 3A = 2(2A - 5) \Leftrightarrow A^2 - 7A + 10 = (A - 5)(A - 2) = 0 \Rightarrow (A, k) = (5,5), (2,-1)$.

Thus, $(R_1, R_2, k) =$ **<u>(5, 2, 5)</u>**, **<u>(2, –1, –1)</u>**.

Alternately, $x^2 - kx + 2k = 2x \Leftrightarrow x^2 - (k+2)x + 2k = (x-k)(x-2) = 0$.

Therefore, the roots are 2 and k, which implies k must be $2 \pm 3 = \begin{cases} -1 \\ 5 \end{cases}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

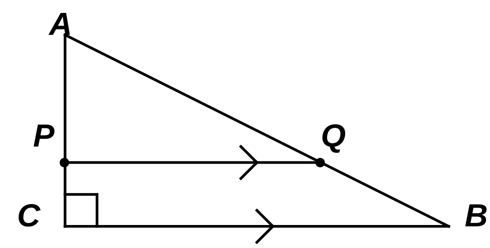
ANSWERS

A) ______________________________

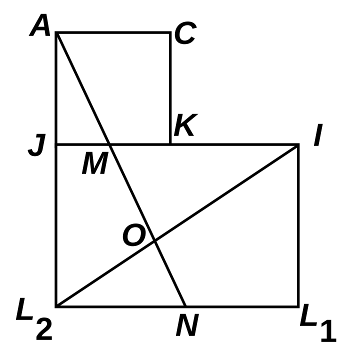
B) _____________ : _______________

C) ______________________________

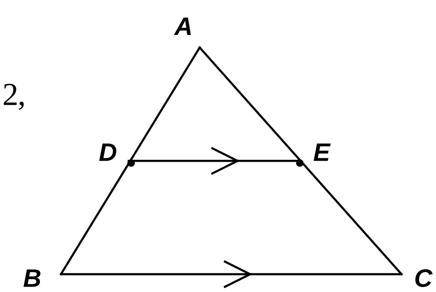
A) Given: Right triangle ABC, with $AB = 17$, $BC = 15$
$\overline{PQ} \parallel \overline{BC}$, $AP : PC = 3 : 1$
Compute the area of the trapezoid $PQBC$.



B) $JACK$ is a square with side 2
JIL_1L_2 is a 3 x 6 rectangle, where $JL_2 < L_2L_1$
Point M is the midpoint of $\overline{JK}$.
Points A, M, O, and N are collinear.
Compute the ratio $\dfrac{NO}{MO}$.

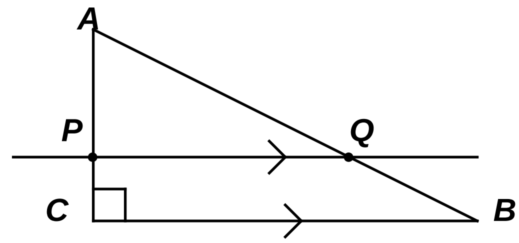


C) In ΔABC, $\overline{DE} \parallel \overline{BC}$, $AD = x + 2$, $AE = \dfrac{x^2}{2}$, $DE = 3x - 2$,
$DB = 2x + 1$ and $EC = x^2 - 4$.
Compute BC.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 SOLUTION KEY

Round 5



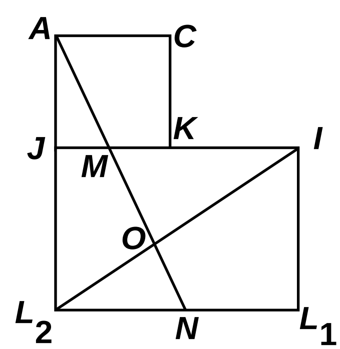
A) $AC^2 = AB^2 - BC^2 = 289 - 225 = 64 \Rightarrow AC = 8$ and the area of ΔABC is $\frac{1}{2} \cdot 8 \cdot 15 = 60$.

$AP : PC = 3 : 1 \Rightarrow AP = 6$

Since $\Delta APQ \sim \Delta ABC$, their areas are in the ratio of the square of their corresponding sides, namely $9:16$. Thus, the area of trapezoid $PQBC$ is $\frac{7}{16}(60) = \frac{7 \cdot 15}{4} = \underline{\mathbf{\frac{105}{4}}}$ or $\underline{\mathbf{26.25}}$.

Alternately, $\Delta APQ \sim \Delta ABC$ with corresponding sides in a ratio of 6 : 8 or 3 : 4

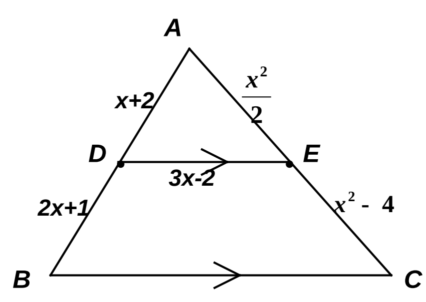
Therefore, $PQ = \frac{3}{4}(15) = \frac{45}{4}$ and the area of the trapezoid is $\frac{1}{2} \cdot 2 \cdot \left(\frac{45}{4} + 15\right) = \underline{\mathbf{\frac{105}{4}}}$.



B) $\Delta JAM \sim \Delta L_2AN \Rightarrow \frac{L_2N}{JM} = \frac{L_2A}{JA} \Rightarrow \frac{L_2N}{1} = \frac{5}{2} \Rightarrow L_2N = 2.5$.

$\begin{cases} L_1L_2 = JK + KI = 6 \\ MK = 1 \end{cases} \Rightarrow MI = 5$.

Since $\Delta MOI \sim \Delta NOL_2$, $\frac{NO}{MO} = \frac{NL_2}{MI} = \frac{2.5}{5} \Rightarrow = \underline{\mathbf{1:2}}$.



C) Since $\Delta ADE \sim \Delta ABC$, $\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC} \Rightarrow \frac{AD}{DB} = \frac{AE}{EC}$

$\Rightarrow \frac{x+2}{2x+1} = \frac{x^2}{2\left(x^2-4\right)} \Rightarrow 2x^3 - 16 + 4x^2 - 8x = 2x^3 + x^2$

$\Rightarrow 3x^2 - 8x - 16 = (3x+4)(x-4) = 0 \Rightarrow x = 4.$

$x = -\frac{4}{3}$ is extraneous. $(\Rightarrow BD = 2x + 1 < 0)$

Thus, $AD = 6$, $DB = 9$, $DE = 10$ and $\frac{6}{6+9} = \frac{10}{BC} \Rightarrow BC = \underline{\mathbf{25}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016
ROUND 6 ALG 1: ANYTHING

ANSWERS

A) ______________________________

B) (________ , ________ , ________)

C) ______________________________

A) Given: $\begin{cases} \dfrac{A}{3} + \dfrac{B}{4} = 10 \\ 3A + 4B = 34 \end{cases}$

Compute $A + B$.

B) $\left(\sqrt{3}+\sqrt{5}+\sqrt{15}\right)\left(\sqrt{3}+\sqrt{5}-\sqrt{15}\right) = a + b\sqrt{c}$, where a, b, and c are integers.

Compute the ordered triple (a, b, c).

C) Two of my favorite flavors of SNAPPLE iced tea mix packets were on sale at Walmart. There were T packets in the case and there were 2.6 times as many Peach as Mango packets. However, at Costco I was able to buy a case containing the <u>same</u> number of packets, but there were 50 more Mango packets in the case, resulting in there being only twice as many Peach as Mango packets. Compute T, the total number of SNAPPLE drink packets in a case.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 SOLUTION KEY

Round 6

A) $\begin{cases} \dfrac{A}{3}+\dfrac{B}{4}=10 \\ 3A+4B=34 \end{cases} \Rightarrow \begin{cases} 4A+3B=120 \\ 3A+4B=34 \end{cases} \Rightarrow 7A+7B=154 \Rightarrow A+B=\underline{\mathbf{22}}$.

It was not necessary to first solve for A and B separately!!

B) Grouping the first terms in each trinomial, we view each trinomial as a binomial and we have the product of a sum and a difference (which is equivalent to the difference of perfect squares!)

$\left(\sqrt{3}+\sqrt{5}+\sqrt{15}\right)\left(\sqrt{3}+\sqrt{5}-\sqrt{15}\right)=\left[\left(\sqrt{3}+\sqrt{5}\right)+\sqrt{15}\right]\left[\left(\sqrt{3}+\sqrt{5}\right)-\sqrt{15}\right]$

$\left(\sqrt{3}+\sqrt{5}\right)^2-\left(\sqrt{15}\right)^2=3+2\sqrt{15}+5-15=-7+2\sqrt{15}\Rightarrow \underline{\mathbf{(-7,2,15)}}$.

C) Let P and M denote the number of Peach and Mango packets purchased. Let $T=P+M$.

$P=2.6M \Rightarrow \dfrac{P}{M}=\dfrac{13}{5}$ and $\dfrac{P-50}{M+50}=\dfrac{2}{1}\Rightarrow P=2M+150$

Substituting, $\dfrac{2M+150}{M}=\dfrac{13}{5}\Rightarrow 10M+750=13M \Rightarrow M=250,\ P=650 \Rightarrow T=\underline{\mathbf{900}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ________________________________ D) ___________________________

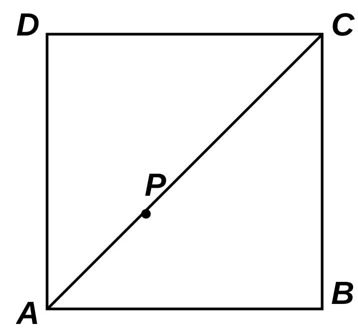
B) { x | ___________________________ } E) ___________________________

C) ________________________________ F) ___________________________

A) Given a hyperbola defined by $3(y+3)^2-(x-4)^2=12$.
The major axis of an ellipse is the transverse (or major) axis of this hyperbola.
(The vertices on the major axis of the ellipse coincide with the vertices of the hyperbola.)
The major axis of the ellipse is twice as long as its minor axis. Compute the length of the line segment on $y=-2$ whose endpoints are on the ellipse.

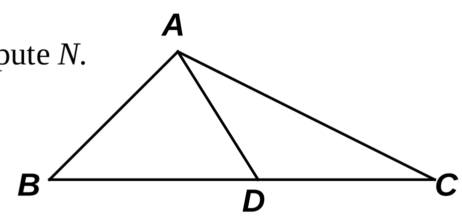
B) $\{(x,y)\,|\,xy(y-2)=x^2-6, \text{ where } y\ge 1\}$ defines a real-valued function $y=f(x)$ whose domain is a subset of the real numbers. Specify the domain of *f*.
For disjoint intervals, the use of the proper connector ("<u>and</u>" or "<u>or</u>") is required.
Alternatively, the symbols "$\wedge$" and "$\vee$" may be used.

C) Compute <u>all</u> possible solutions *x*, where $x\in[0,\pi]$, to the equation $\tan^2 x+\cot^2 x=14$.



D) Given: *ABCD*, a square with side 6.
Beetle Bailey received these marching orders:
- Start at *A*.
- Proceed along $\overline{AC}$.
- Stop at point *P* which is twice as far from *B* as it is from *A*.

These instructions are way above Beetle's pay grade, but with your help he can prove how smart he is to his commanding officer. In simplified form, $AP=\sqrt{2}\left(\sqrt{N}-1\right)$, where *N* is an integer. Compute *N*.



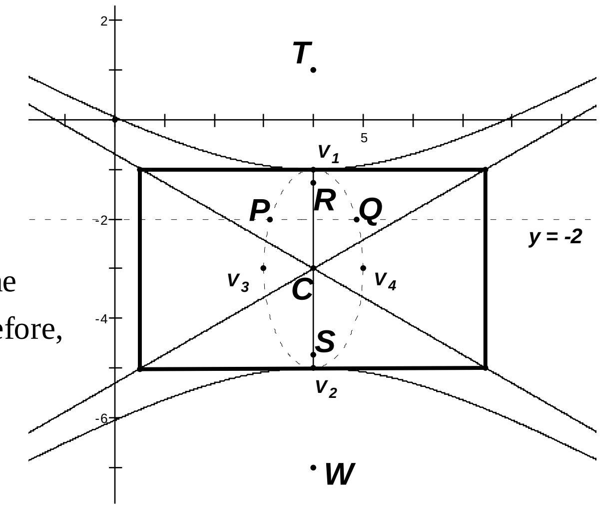
E) Given: $m\measuredangle DAC=m\measuredangle DCA,\ \Delta BAC\sim\Delta BDA,\ AB=1,\ AC=2$.
Compute *AD*.

F) My little brother JJ and I are very competitive. JJ's best mile time to date is 4:30.
JJ's goal is to be able to beat my time. Assuming he improves his time *k*% each year for the next two years, he will beat my current best time of 4:03.
Determine the <u>smallest integer</u> value of *k* which allows him to reach his goal.
[A helpful fact: $\sqrt{10}\approx 3.162$]

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 SOLUTION KEY

Team Round

A) $3\left(y+3\right)^2-\left(x-4\right)^2=12 \Leftrightarrow \dfrac{\left(y+3\right)^2}{4}-\dfrac{\left(x-4\right)^2}{12}=1$



This is a vertical hyperbola with center at $C\left(4,-3\right)$, $a=2, b=2\sqrt{3}$ and $c=4$.
Thus, the vertices of the hyperbola are at $\left(4,-3+2\right)=V_1\left(4,-1\right)$ and $\left(4,-3-2\right)=V_2\left(4,-5\right)$

The center of the ellipse is $\left(4,-3\right)$, $a=2, b=1$ (since the major axis is twice as long as the minor axis) and, therefore, its equation is $\dfrac{\left(y+3\right)^2}{4}+\left(x-4\right)^2=1$.

Substituting, when $y=-2$, we have

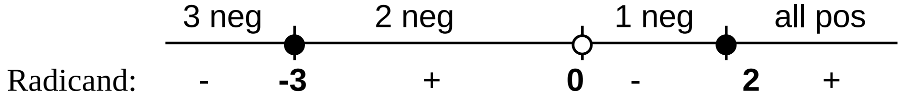
$\dfrac{1}{4}+\left(x-4\right)^2=1 \Rightarrow \left(x-4\right)^2=\dfrac{3}{4} \Rightarrow x=4\pm\dfrac{\sqrt{3}}{2}$ and the length of the segment $\overline{PQ}$ is $\left(4+\dfrac{\sqrt{3}}{2}\right)-\left(4-\dfrac{\sqrt{3}}{2}\right)=\underline{\boldsymbol{\sqrt{3}}}$.

FYI: R and S are the foci of the ellipse, while T and W are the foci of the hyperbola. V_3 and V_4 are the endpoints of the minor axis of the ellipse.
You should be able to specify the coordinates of these points, as well as the equations of the lines containing the diagonals of the rectangle which are referred to as the asymptotes of the hyperbola. Although neither branch of the hyperbola ever intersects these lines, as $x\to\pm\infty$, the distance between the hyperbola and these lines becomes arbitrarily small.

B) $xy\left(y-2\right)=x^2-6 \Rightarrow y^2-2y=\dfrac{x^2-6}{x} \Rightarrow \left(y-1\right)^2=\dfrac{x^2-6}{x}+1=\dfrac{x^2+x-6}{x}=\dfrac{\left(x+3\right)\left(x-2\right)}{x}$. Thus,

$y=1\pm\sqrt{\dfrac{\left(x+3\right)\left(x-2\right)}{x}}$ and since it was given that $y\geq 1$, we have $y=f\left(x\right)=1+\sqrt{\dfrac{\left(x+3\right)\left(x-2\right)}{x}}$.

The critical values are $-3, 0, 2$ and, as each critical value is passed from left to right, there is one less negative factor in the radicand. On the left (as $x\to-\infty$) all three factors are negative; whereas, on the right (as $x\to+\infty$) all three factors are positive. The following chart summarizes this discussion.



Thus, the domain is $\underline{\left\{x \mid \boldsymbol{-3\leq x<0} \text{ or } \boldsymbol{x\geq 2}\right\}}$. The symbol "$\vee$" may be used in place of "or".

Team Round - continued

C) $\tan^2 x+\cot^2 x=14 \Leftrightarrow \dfrac{\sin^2 x}{\cos^2 x}+\dfrac{\cos^2 x}{\sin^2 x}=14 \Leftrightarrow \sin^4 x+\cos^4 x=14\sin^2 x\cos^2 x$

$\Leftrightarrow \sin^4 x+2\sin^2 x\cos^2 x+\cos^4 x=16\sin^2 x\cos^2 x \Leftrightarrow \left(\sin^2 x+\cos^2 x\right)^2=16\sin^2 x\cos^2 x$

$\Leftrightarrow 16\sin^2 x\cos^2 x=1$. Dividing through by 4 and applying the double angle identity, $\sin 2\theta=2\sin\theta\cos\theta$, $4\sin^2 x\cos^2 x=\left(2\sin x\cos x\right)^2=\left(\sin 2x\right)^2=\dfrac{1}{4} \Leftrightarrow \sin 2x=\pm\dfrac{1}{2}$.

Thinking the 30°-family of related angles in all 4 quadrants , i.e. $\dfrac{\pi}{6}$ radians, we have

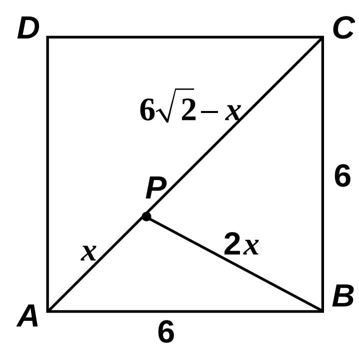
$$2x=\begin{cases}\dfrac{\pi}{6}+n\pi \ (\text{quadrants } 1,3)\\ \dfrac{5\pi}{6}+n\pi \ (\text{quadrants } 2,4)\end{cases} \Rightarrow x=\begin{cases}\dfrac{\pi}{12}+\dfrac{n\pi}{2}\\ \dfrac{5\pi}{12}+\dfrac{n\pi}{2}\end{cases} \Rightarrow x=\dfrac{\pi}{12}\begin{cases}1+6n\\ 5+6n\end{cases}.$$

For $n=0,1$, we have $x=\underline{\boldsymbol{\dfrac{\pi}{12},\dfrac{5\pi}{12},\dfrac{7\pi}{12},\dfrac{11\pi}{12}}}$.

Alternately, $\tan^2 x+\cot^2 x=14 \Leftrightarrow \tan^2 x+\cot^2 x+2=16$

$$\Leftrightarrow \left(\tan x+\frac{1}{\tan x}\right)^2=\frac{\left(\tan^2 x+1\right)^2}{\tan^2 x}=\frac{\sec^4 x}{\tan^2 x}=\frac{1}{\cos^4 x}\cdot\frac{\cos^2 x}{\sin^2 x}=\frac{1}{\left(\sin x\cdot\cos x\right)^2}=16$$

$\Leftrightarrow \dfrac{4}{\left(2\sin x\cos x\right)^2}=\dfrac{4}{\sin^2 2x}=16$. Thus, $\sin^2 2x=\dfrac{1}{4}$ and the same result follows.



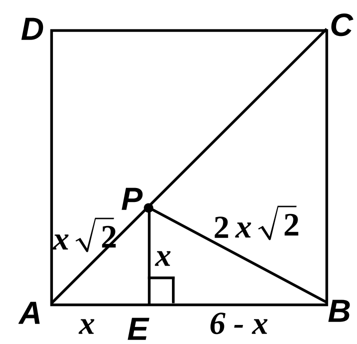
D) Using the dimensions in the diagram at the right, apply Stewart's Theorem. $6^2\cdot x+6^2\cdot\left(6\sqrt{2}-x\right)=\left(2x\right)^2\cdot 6\sqrt{2}+x\cdot\left(6\sqrt{2}-x\right)\cdot 6\sqrt{2}$

$\Rightarrow 36x+216\sqrt{2}-36x=24\sqrt{2}x^2+72x-6\sqrt{2}x^2$

$\Rightarrow 18\sqrt{2}x^2+72x-216\sqrt{2}=0$

$\Rightarrow \sqrt{2}x^2+4x-12\sqrt{2}=0$. Applying the quadratic formula,

$$x=AP=\frac{-4+\sqrt{16+96}}{2\sqrt{2}}=\frac{-4+4\sqrt{7}}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\left(-1+\sqrt{7}\right)\sqrt{2} \Rightarrow N=\underline{7}.$$

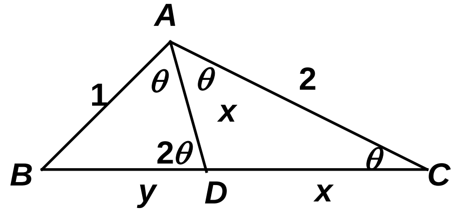


Alternately, without resorting to Stewart's Theorem, drop a perpendicular from P to $\overline{AB}$. Let $AE=x$ and mark the other sides as indicated. In ΔBEP, $x^2+\left(6-x\right)^2=\left(2x\sqrt{2}\right)^2$

$\Rightarrow 2x^2-12x+36=8x^2 \Rightarrow x^2+2x-6=0 \Rightarrow x=-1+\sqrt{7}$

$\Rightarrow AP=\left(-1+\sqrt{7}\right)\sqrt{2} \Rightarrow N=\underline{7}$.

Team Round - continued



E) Let $DC = x$, $BD = y$ and $m\measuredangle DAC = m\measuredangle DCA = \theta$.
As an exterior angle of ΔADC, $m\measuredangle BDA = 2\theta$
$m\measuredangle DAC = m\measuredangle DCA \Rightarrow DA = DC$.
$\Delta BAC \sim \Delta BDA \Rightarrow m\measuredangle BCA = m\measuredangle BAD = \theta$. Thus, $\overrightarrow{AD}$ is an angle bisector and $\frac{y}{1} = \frac{x}{2} \Rightarrow x = 2y$. $\Delta BAC \sim \Delta BDA \Rightarrow \frac{BC}{BA} = \frac{AC}{DA} \Leftrightarrow \frac{x+y}{1} = \frac{2}{x}$.

Cross multiplying, $x^2 + xy = 2 \Leftrightarrow 4y^2 + 2y^2 = 2 \Leftrightarrow y^2 = \frac{1}{3} \Rightarrow y = \frac{\sqrt{3}}{3} \Rightarrow AD = x = \underline{\mathbf{\frac{2\sqrt{3}}{3}}}$.

F) JJ's current mile time is 270 seconds; big brother's is 243. After one year, JJ's time will be $270(1-k\%) = 270\left(\frac{100-k}{100}\right)$. After 2 years, his time will be

$\left(270\left(\frac{100-k}{100}\right)\right)\left(\frac{100-k}{100}\right) = 270\left(\frac{100-k}{100}\right)^2$. We require that

$270\left(\frac{100-k}{100}\right)^2 < 243 \Rightarrow (100-k)^2 < \frac{243}{270}\cdot 10^4 = 9\cdot 10^3 \Rightarrow (100-k) < \sqrt{9\cdot 10^3} = 30\sqrt{10}$

$k > 100 - 30\sqrt{10} \approx 100 - 94.86 \approx 5.14$. Therefore, k must be at least **<u>6</u>**.

FYI: A 6% improvement over 2 years would drop little brother's time to 238.57 seconds and he would break the 4:00 mile (240 seconds), whereas a 5% improvement would result in a mile-time of 243.675 seconds and he would fall just short of beating big brother.

Some real world perspective:
The first American to run a sub 4-minute mile was Don Bowdon in 1957 (3:58.7). Jim Ryan was the first American <u>high school runner</u> to break the 4-minute mile (3:55.3 in 1965 - Witchita, KS). 5 others have joined him since - Tim Danielson (1966), Marty Liquori (1967), Alan Webb (2001), Lukas Verzbicas (2011), and Matthew Maton (2015). Only Webb has had a faster time (3:53.43). The current world record time for the mile of 3:43.13 (set in 1999 in Rome) belongs to Morocco's Hicham El Guerrouj who bested Kenyan Noah Ngeny's time of 3:43.40 **in the same race** with quarter-mile splits of 55.6, 56.0, 56.3 and 55.2 . Two runners bettered the then world record of Algerian Noureddine Morceli (3:44.39) by almost a full second! As of this date (2016), these three times are still the fastest mile times ever recorded! You can watch the 1999 record setting race on Youtube. The link is https://www.youtube.com/watch%3Fv%3DJi0yK7fV5Rk.

Roger Bannister
First to break 4-minute barrier
(May 6th, 1954 3:59.4)

Hicham El Guerrouj bests Noah Ngeny by
0.27 seconds in a world-record time of 3:43.13
July 7th, 1999

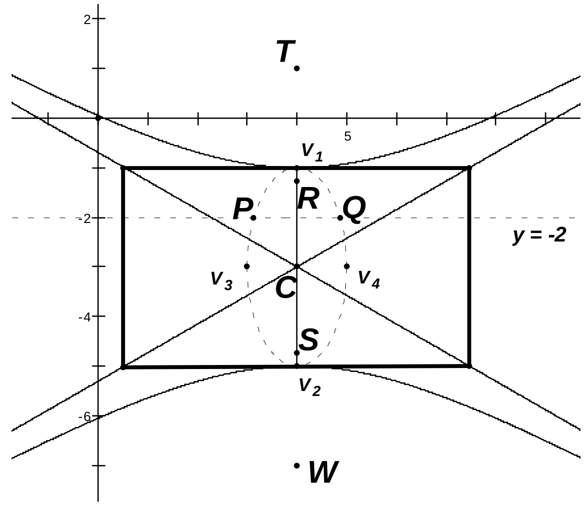
Team Question A: Center $C(4,-3)$

Since $a=2, b=2\sqrt{3}$ and $c=4$ for the hyperbola, and $a=2, b=1$ and $c=\sqrt{3}$ for the ellipse.

$R\left(4, -3+\sqrt{3}\right)$, $S\left(4, -3-\sqrt{3}\right)$

$T(4, 1)$, $W(4, -7)$

$V_3(3, -3)$, $V_4(5, -3)$



Since the slope of asymptotes are $\pm\frac{2}{2\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$,

the point-slope equations of the asymptotes are $(y+3) = \pm\frac{\sqrt{3}}{3}(x-4)$.

In the diagram below, graphically, it certainly appears as if the hyperbola approaches arbitrarily close to these lines. Analytically, why is this the case?

Staring with $3(y+3)^2-(x-4)^2=12 \Leftrightarrow \frac{(y+3)^2}{4}-\frac{(x-4)^2}{12}=1$, solve for *y*.

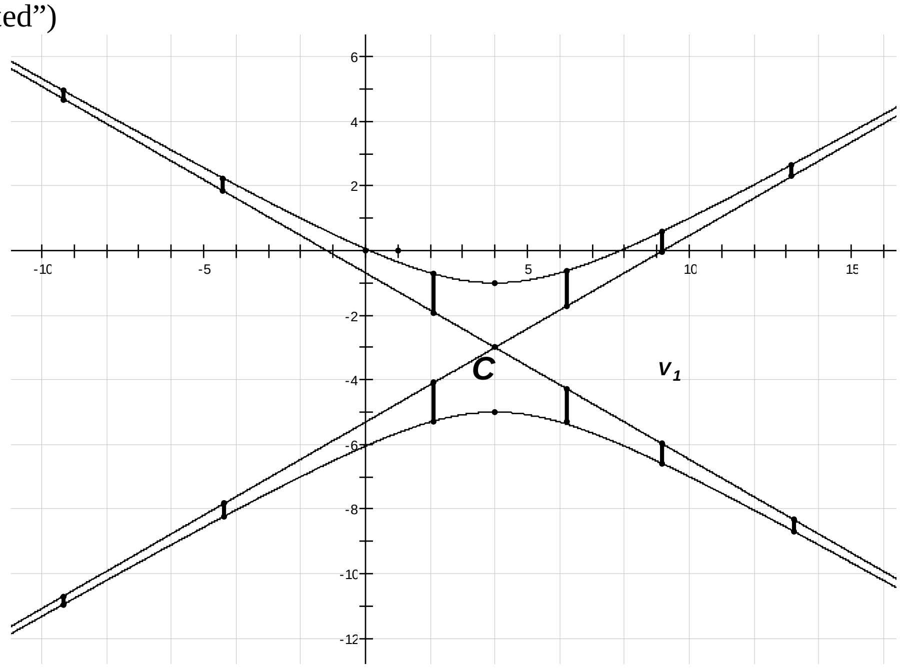
$$(y+3)^2 = \frac{(x-4)^2}{3}+4$$

As $x \to \pm\infty$, that is, as *x* becomes an arbitrarily large positive number (moving to the right on the graph) or an arbitrarily small negative number (moving to the left on the graph), the 4 becomes *negligible* and we have

$$(y+3)^2 \approx \frac{(x-4)^2}{3} \Leftrightarrow y+3 \approx \pm\frac{1}{\sqrt{3}}(x-4)$$

Thus, as $x \to \pm\infty$, the *y*-coordinate of a point on the straight lines (the asymptotes) becomes an increasingly better approximation of the *y*-coordinate of the corresponding point on corresponding branch of the hyperbola.

Q.E.D (‘Quod erat demonstratum’, Latin for “enough said”, literally “that which was to be demonstrated”)



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2016 ANSWERS

Round 1 Analytic Geometry: Anything

A) 19 B) (16, 2) C) $-\frac{8\sqrt{7}}{3}$

Round 2 Alg: Factoring

A) $3, -\frac{7}{2}$ B) $(19,17), (11,7), (9,3)$ in any order C) $(x^2+y^2)(x-1)(x+1)$ or $(x^2+y^2)({}^{-}1+x)(1+x)$ in any order

Round 3 Trig: Equations

A) π B) 330 C) 8

Round 4 Alg 2: Quadratic Equations

A) 361 B) –64 C) (5, 2, 5) and (2, −1, −1)

Round 5 Geometry: Similarity

A) 26.25 or $\left(\frac{105}{4}\right)$ B) 1 : 2 C) 25

Round 6 Alg 1: Anything

A) 22 B) $(-7,2,15)$ C) 900

Team Round

A) $\sqrt{3}$

B) $-3 \le x < 0$ or $x \ge 2$

C) $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

D) 7

E) $\frac{2\sqrt{3}}{3}$

F) 6