MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015
ROUND 1 VOLUME & SURFACES

ANSWERS

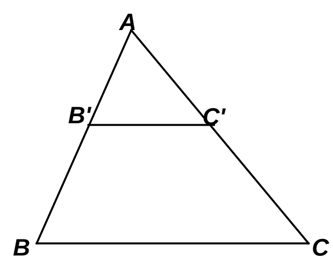
A) ______________________________

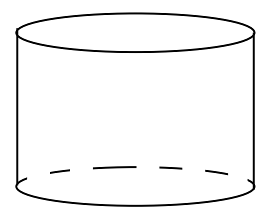
B) ______________ : ______________

C) ______________________________

A) A sphere with <u>integer</u> radius r has a volume greater than a cube with a side of length 8. Determine the <u>smallest</u> possible value of r.
Assume $\pi \approx 3.1416$.

B) Given: $\overline{B'C'} \parallel \overline{BC}$ and $B'C' = \frac{1}{3}BC$
The radius of the base of a right circular cylinder is $\frac{7}{8}$ of the altitude of $\Delta AB'C'$. The base of $\Delta AB'C'$ is $\frac{5}{4}$ of the radius of the right circular cylinder.





Find the ratio of the area of a base of the cylinder to the area of ΔABC.

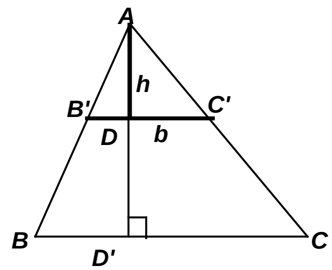
C) Assume an underground storage tank is a cylinder 72 inches long and 36 inches wide, capped on each end by a hemisphere of radius 18 inches. Assume there are exactly 7.5 gallons of water in a cubic foot. In terms of π, compute the <u>maximum</u> number of gallons of water this tank will hold. Leave your answer in terms of π.

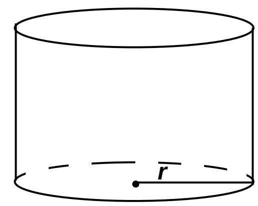
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015 SOLUTION KEY

Round 1

A) The volume of the cube is $8^3 = 512$.
A sphere of radius 4 could be inscribed in the cube of side 8 and, therefore, would have a smaller volume. Trying a radius of 5,
$V_{sphere} = \frac{4}{3}\pi \cdot 5^3 = \frac{500\pi}{3} = \frac{1000\pi}{6} \approx \frac{1000(3.1416)}{6} = \frac{3141.6}{6} = 523^+ > 512$. Thus, $r = \underline{\mathbf{5}}$.

B) $\Delta AB'C' \sim \Delta ABC$ with a ratio of similitude of $\frac{1}{3}$.





Let $h = AD$ and $b = B'C'$.
$r = \frac{7}{8}h \Rightarrow AD' = 3h = 3\left(\frac{8}{7}r\right) = \frac{24r}{7}$
$b = \frac{5}{4}r \Rightarrow BC = 3b = \frac{15r}{4}$ Thus,
the area(circle) = πr^2 and area(ΔABC) = $\frac{1}{2}\left(\frac{24}{7}r\right)\left(\frac{15}{4}r\right) = \frac{45}{7}r^2 \Rightarrow \underline{\boldsymbol{\frac{7\pi}{45}}}$.

C) The volume of the tank is the volume of a cylinder plus the volume of a sphere.
$V = \pi \cdot 18^2 \cdot 72 + \frac{4}{3} \cdot \pi \cdot 18^3 = 18^3 \cdot \pi \cdot \left(4 + \frac{4}{3}\right)$ inches^3.

Converting to cubic feet, we have $\dfrac{18^3 \cdot \pi \cdot \left(4 + \frac{4}{3}\right)}{12^3} = \left(\frac{3}{2}\right)^3 \cdot \pi \cdot \frac{16}{3} = 18\pi$.

Converting to gallons, we have $18\pi \cdot \frac{15}{2} = \underline{\mathbf{135\pi}}$ gallons.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Right triangle *ABC* has legs of length 8 and 15.
$\Delta DEF \sim \Delta ABC$ and the sides of ΔDEF have integer lengths. If the perimeter of ΔDEF is greater than 2^{10}, compute the minimum length of the hypotenuse of ΔDEF.

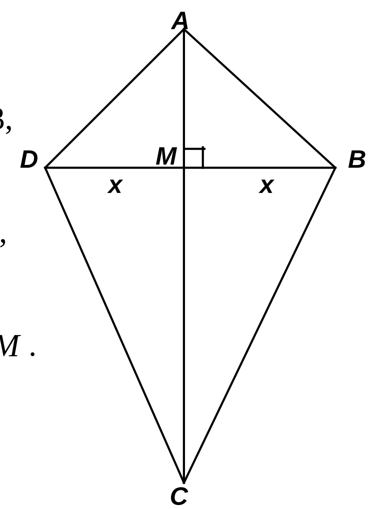
B) A kite has diagonals of lengths 18 and 52.
Its sides also have integer lengths. The perimeter of the kite is 112.
The short diagonal divides the long diagonal into a reduced ratio of $a : b$, where $a > b$. Compute $a + b$.

C) In right triangle *ABC*, $AB = 2\sqrt{k}$, $AC = \sqrt{30}$, and $BC = k\sqrt{2}$.
Compute all possible areas of ΔABC.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015 SOLUTION KEY

Round 2

A) The hypotenuse of ΔABC has length 17 $\left(8^2+15^2=289=17^2\right)$, producing a perimeter of 40.
Since the sides of similar triangles (and hence the perimeters) are proportional, we have $40k > 2^{10} = 1024 \Rightarrow k \geq 26$.
Thus, the minimum length of the hypotenuse of ΔDEF is $26 \cdot 17 = \underline{\mathbf{442}}$.



B) Since either 2*x* = 18 or 2*x* = 52, a leg of a right triangle *BAM* must be either 9 or 26. Listing the common right triangles, we have 3-4-5, 5-12-13, 7-24-25, 9-40-41, 11-60-61, $\boxed{\text{13-84-85}}$, …
If 26 were a <u>leg</u> of one of the right triangles, the lengths of two sides of the kite would be 170 and our perimeter would already exceed 112. Therefore, we are sure that the legs on the short diagonal must both be 9.
We try 3(3-4-5) = 9-12-$\boxed{15}$ as the dimensions of ΔBAM and ΔDAM.
112 – 30 = 82 This results in dimensions of 9-40-41 for ΔCDM and ΔCBM.
Thus, the long diagonal is divided into segments of lengths 12 and 40.
$\frac{40}{12} = \frac{10}{3} \Rightarrow \underline{\mathbf{13}}$.

C) Given: Right triangle *ABC*, with sides of length $AB = 2\sqrt{k}$, $AC = \sqrt{30}$, and $BC = k\sqrt{2}$.
Case 1: $\overline{AB}$ is the hypotenuse.
$4k = 30 + 2k^2 \Leftrightarrow k^2 - 2k + 15 = 0$ which has no solutions over the real numbers.

Case 2: $\overline{AC}$ is the hypotenuse
$30 = 4k + 2k^2 \Leftrightarrow k^2 + 2k - 15 = (k-3)(k+5) = 0 \Rightarrow k = 3$, producing an area of
$\frac{1}{2} \cdot 2\sqrt{3} \cdot 3\sqrt{2} = \underline{\mathbf{3\sqrt{6}}}$.
Case 3: $\overline{BC}$ is the hypotenuse
$4k + 30 = 2k^2 \Leftrightarrow k^2 - 2k - 15 = (k-5)(k+3) = 0 \Rightarrow k = 5$, producing an area of
$\frac{1}{2} \cdot 2\sqrt{5} \cdot \sqrt{30} = \underline{\mathbf{5\sqrt{6}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) (_____________ , _____________)

A) The equation $2(Ax+2)=3-(Bx-1)+x$ is an identity, that is, an equation true for <u>all</u> values of x. Compute the value of $2A + B + 3$.

B) $\frac{5}{8}$ of the students attending a school function are girls and $\frac{2}{5}$ of the students attending this school function like to dance. After these students are joined by 20 more girls from another school, all of whom like to dance, 75% of the students in attendance are girls. k percent of the students now at this school function like to dance. Compute k.

C) The cost of a wedding banquet is the sum of a fixed cost F, which is the same regardless of how many people attend, and a variable cost V, which depends on the number of people attending. Assuming a charge of d dollars per person, the total cost of a banquet attended by 75 people is \$4400, while the total cost of a banquet attended by 400 people is \$18,050. Compute the ordered pair (F,d).

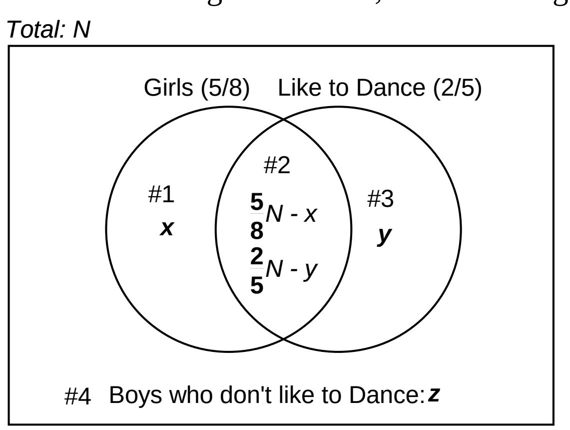
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015 SOLUTION KEY

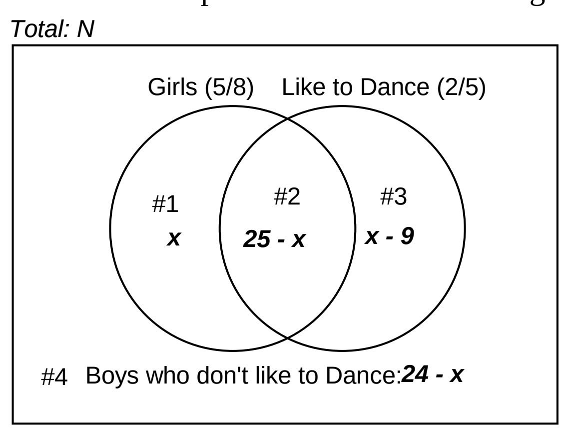
Round 3

A) $2(Ax+2)=3-(Bx-1)+x \Leftrightarrow 2Ax+4=4+(1-B)x$
This is true for all x if (and only if) $2A = 1 - B$ or $2A + B = 1$.
Thus, $2A + B + 3 =$ **4**.

B) Suppose the original attendance is N students. Then:
$\frac{5}{8}N+20=\frac{3}{4}(N+20) \Leftrightarrow 5N+160=6N+120 \Rightarrow N=40$
If $\frac{2}{5}$ of these students like to dance, then the total number of dancers is $\frac{2}{5}\cdot 40+20=36$
and $\frac{36}{60}=\frac{6}{10}=\frac{60}{100} \Rightarrow k=\underline{\mathbf{60}}$.
Note: There would be multiple solutions to questions like:
x - How many girls from the original school did not like to dance?
y - How many boys from the original school liked to dance?
z - How many boys from the original school did not like to dance?

After determining that $N = 40$, the Venn diagram on the left simplifies to the one on the right.





We see that $9 \le x \le 24$ and 16 different ordered triples (x, y, z) satisfy the conditions of the original problem and all of them give us $k = 60$.

C) $\begin{cases} F+75d=4400 \\ F+400d=18050 \end{cases} \Rightarrow F=4400-75d=18050-400d \Rightarrow 325d=13650 \Rightarrow d=42$
$(F,d)=\underline{(\mathbf{1250,42})}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

ANSWERS

A) ______________________________

B) ______________________________

C) (_____________ , _____________)

A) For certain values of the integer constants *A* and *Y*, $\frac{21-A}{52}=\frac{Y}{4}$. If $\frac{Y}{4}$ is a positive proper fraction in simplified form, compute all possible ordered pairs (*A*, *Y*).

B) $\frac{20x+k}{2x^2-x-3}=\frac{A}{2x-3}+\frac{B}{x+1}$ is an identity in x (*A*, *B* and *k* are constants).
Compute *k*, if $A:B=4:3$.

C) While making a long drive on Interstate I-95, Dick occasionally checked his odometer. It read 12345 at 1:10 PM and 126♦1 at 5:55 PM. The tens digit *d* was obscured by an annoying insect (♦). If Dick was driving at a steady speed of k mph, for some integer *k*, compute the ordered pair (k,d).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015 SOLUTION KEY

Round 4

A) Since $\frac{Y}{4}$ is a completely reduced proper fraction for $Y > 0$, Y = 1 or 3.

Thus, $(21 - A)\cdot 4 = 52Y \Rightarrow 21 - A = 13Y = 13$ or $39 \Rightarrow A = 8, -18 \Rightarrow (A, Y)$ = **<u>(8, 1)</u>**, **<u>(–18, 3)</u>**.

B) $\frac{A}{2x-3} + \frac{B}{x+1} = \frac{A(x+1)+B(2x-3)}{(2x-3)(x+1)} = \frac{(A+2B)x+(A-3B)}{2x^2-x-3} = \frac{20x+k}{2x^2-x-3}$

This is an identity in x if and only if $A + 2B = 20$ and $k = A - 3B$.
$A : B = 4 : 3 \Rightarrow A = 4C, B = 3C$

Now $A + 2B = 10C = 20 \Rightarrow C = 2 \Rightarrow A = 8, B = 6$ and $k = A - 3B = 8 - 3(6) = \underline{\mathbf{-10}}$.

Alternate Solution:

Using $\frac{A}{B} = \frac{4}{3}$, after expressing the sum over a common denominator, we equate numerators,

$$A(x+1) + \left(\frac{3A}{4}\right)(2x-3) = 20x + k \text{.} \quad \begin{cases} x = -1 \Rightarrow -\frac{15A}{4} = -20 + k \\ x = 0 \Rightarrow -\frac{5A}{4} = k \end{cases} \Rightarrow -20 + k = 3k \Rightarrow k = \underline{\mathbf{-10}}$$

C) Dick travelled (126*d*1 – 12345) in 4 hours and 45 minutes.
$(600 + 10d + 1) - 345 = 256 + 10d$ miles

His average speed was $k = \frac{256+10d}{4.75} = \frac{4(256+10d)}{19} = \frac{8(128+5d)}{19}$ mph.

For $d = 1$, we have $k = \frac{8(133)}{19} = 8 \cdot 7 = 56 \Rightarrow (k, d) = \underline{(\mathbf{56,1})}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A) $k =$ ______________________________

B) ______________________________

C) ______________________________

A) For what integer k does the inequality $0 < |x| < k$ have <u>exactly</u> 20 integer solutions?

B) Solve for x over the reals. $\left| x - |2x+3| \right| = 1 - x$

C) Solve for x over the reals. $\dfrac{2}{x+6} - \dfrac{3}{5-x} \geq 0$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015 SOLUTION KEY

Round 5

A) For integer values of k, $0<|x|<k \Leftrightarrow 0<|x|\le k-1$. As long as $k-1$ is positive, there are $k-1$ positive solutions and $k-1$ negative solutions. $2(k-1)=20 \Rightarrow k=\underline{\mathbf{11}}$.

B) $\left| x-\left|2x+3\right| \right| = 1-x$

Since any solution requires that the right hand side of the equation be nonnegative, we have a <u>pre-condition</u> that $1-x\ge 0 \Leftrightarrow x\le 1$.

Case 1: $x-|2x+3| = +(1-x) \Rightarrow |2x+3| = 2x-1$

Now we have $\frac{1}{2}\le x\le 1 \Rightarrow 1\le 2x\le 2 \Rightarrow \begin{cases} 4\le 2x+3\le 5 \\ 0\le 2x-1\le 1 \end{cases}$

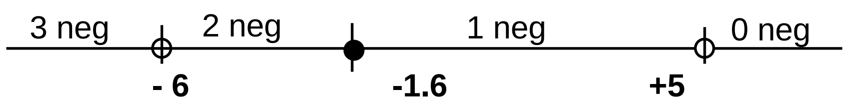
Since these intervals do not overlap, this case produces no solutions.

Case 2: $x-|2x+3| = -(1-x) \Rightarrow |2x+3| = 1$

$\Rightarrow 2x+3=\pm 1 \Leftrightarrow x=\underline{\mathbf{-1,-2}}$ (Since both of these values satisfy the pre-condition.)

C) $\dfrac{2}{x+6}-\dfrac{3}{5-x}\ge 0 \Leftrightarrow \dfrac{2}{x+6}+\dfrac{3}{x-5}\ge 0 \Leftrightarrow \dfrac{2(x-5)+3(x+6)}{(x+6)(x-5)}\ge 0 \Leftrightarrow \dfrac{(5x+8)}{(x+6)(x-5)}\ge 0$

The critical values are –6, –8/5 and +5.
The sign of the quotient depends of how many of the three binomials return a positive value.
This information is summarized in the following diagram.



Thus, we have $\underline{\boldsymbol{-6 < x \le -\frac{8}{5}} \textbf{ or } \boldsymbol{x > 5}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) A uniform 3 foot wide walkway surrounds a rectangular garden with integer dimensions W and L, where $W > L$. Compute the number of distinct ordered pairs (W, L) for which the area of the walkway is 210 square feet.

B) Determine how many factors of $10^4 - 5^4$ fit in at least one of these categories:

- multiple of 5
- nonzero power of 5.

C) Compute $(1204)_8 + (245)_6 - (444)_5$ and express your answer in base 10.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015 SOLUTION KEY

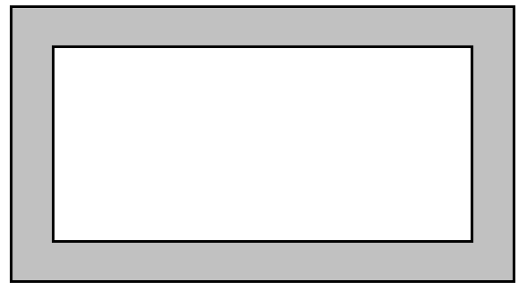
Round 6

A) $(W+6)(L+6)-WL=6W+6L+36=210 \Leftrightarrow W+L+6=35 \Leftrightarrow W+L=29$

Since $W>L$, the rectangle is not a square.

$\Rightarrow (W,L)=(28,1),\ (27,2),\ (26,3),\ ...,\ (15,14)$

a total of <u>**14**</u> ordered pairs.



B) $10^4-5^4=(10^2+5^2)(10^2-5^2)=125(10+5)(10-5)=5^3\cdot(3\cdot5)\cdot5=3^1\cdot5^5$

This product has $(1+1)(5+1)=12$ factors.

1 is not a multiple of 5 and $1=5^0$ is not a nonzero power of 5, but all other factors (except 3) are multiples of 5. Therefore, we have exactly <u>**10**</u> factors that satisfy at least one of the given conditions. Note that the factors like 5, 25, 125, 625 and 3125 satisfy both criterion.

C) $(1204)_8=8^3+2\cdot8^2+4=2^9+2^7+4=512+132=644$

$(245)_6=2\cdot6^2+4\cdot6+5=72+29=101$

Since $444_5+1=1000_5$, $444_5=(1000)_5-1=5^3-1=125-1=124$

$644+101-124=\underline{\mathbf{621}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015
ROUND 7 TEAM QUESTIONS

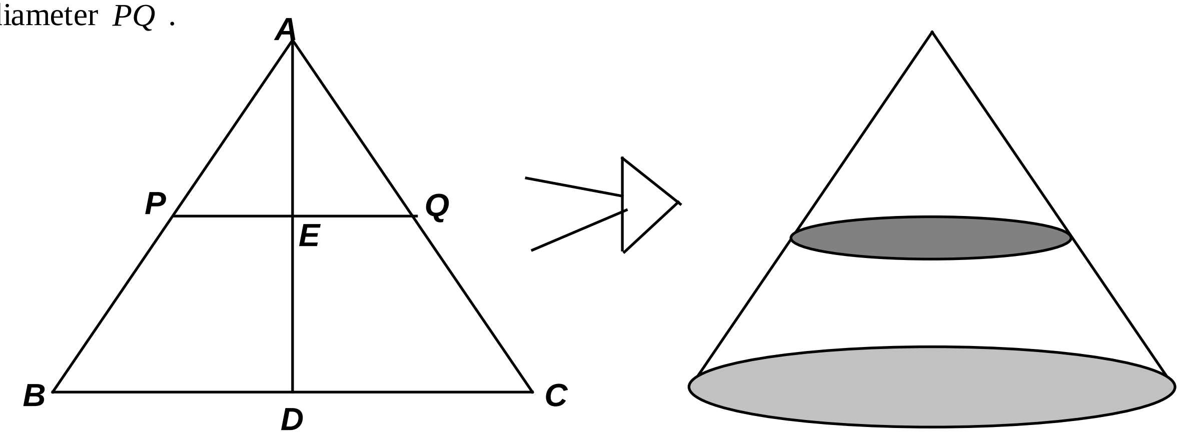
ANSWERS

A) ____________________________ D) (____________ , ____________)

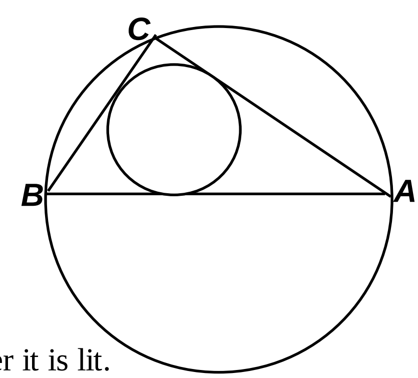
B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

A) Rotating isosceles ΔABC and segment $\overline{PQ}$ about altitude $\overline{AD}$ produces a cone which is subdivided into a smaller cone and the frustum of a cone by a circular cross section with diameter $\overline{PQ}$.



If $AB = AC = 35$ and $BC = 42$, compute the length of $\overline{AE}$ for which the ratio of the volume of the small cone to the volume of frustum is $27:98$.



B) In right ΔABC, $m\angle C = 90°$, the <u>radius</u> of its inscribed circle is $x - y$, the <u>diameter</u> of its circumscribed circle is $x + 5y$, $AC = x + 3y + 1$, and $BC = 2x + y - 3$. Compute the perimeter of ΔABC .

C) Assume an inexpensive candle is 16 inches long and lasts 4 hours after it is lit.
Assume an expensive candle is 12 inches long and lasts 6 hours after it is lit.
Both candles are lit simultaneously.
After S minutes, one candle is twice as long as the other.
After T minutes, the sum of the lengths of the two candles is 10 inches.
Compute $T - S$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015
ROUND 7 TEAM QUESTIONS

D) A projectile is fired at a target 1 mile away (i.e. 5280 feet), but it never reaches the target. Every second it moves to a point which is half the distance to the target.
(After 1 second, it has moved 2640 feet.)
After a minimum of k seconds, it is less than 88 feet from the target and, to the nearest integer, its average speed over k seconds must be at least S ft/sec.
Compute the ordered pair (k, S).

E) A highway department truck, traveling at a constant speed, is spraying the center line on a newly paved highway, where passing is permitted. The spray gun is supposed to cycle on for 1 second and off for 3/10 of a second. Due to a timing malfunction, the time for the off cycles is uniformly increasing 3/10, 4/10, 5/10, … . Instead of the expected uniform striping

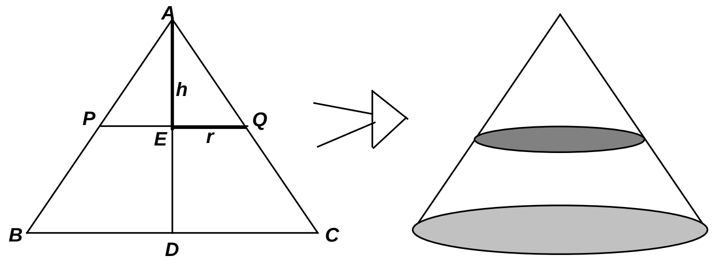
we get

as the gaps between strips get increasingly longer. The distance between points A and B contains 4 complete stripes. If $AB = 104$ feet, compute how many complete or partial stripes will be painted in the first mile (5280 feet) of this new road.

F) Lattice points in the interior of the rectangular region in quadrant 1 with a pair of opposite vertices at the origin and $(N, 3)$ determine 450 segments whose lengths are greater than 1 unit.
Compute N.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015 SOLUTION KEY

Team Round

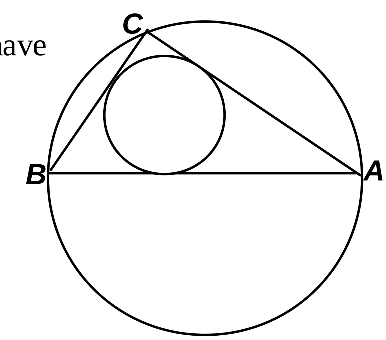


A) $AC = 35$ and $BD = CD = 21$
$\Rightarrow$ $AD = 28$ [7(3 - 4 - 5)].
Let $(AE, QE) = (h, r)$.
By similar triangles, $\dfrac{h}{r} = \dfrac{28}{21} = \dfrac{4}{3} \Rightarrow r = \dfrac{3}{4}h$.
The volume of the frustum is just the difference in the volumes of two cones.

$$\frac{\frac{1}{3}\pi\left(\frac{3}{4}h\right)^2 h}{\frac{1}{3}\pi(21)^2(28)-\frac{1}{3}\pi\left(\frac{3}{4}h\right)^2 h} = \frac{\cancel{\frac{\pi}{3}}\left(\frac{9}{16}\right)h^3}{\cancel{\frac{\pi}{3}}\left((21)^2(28)-\left(\frac{9}{16}\right)h^3\right)} = \frac{9h^3}{K-9h^3} = \frac{a}{b} = \frac{27}{98}$$, where $K = 21^2 \cdot 28 \cdot 16$.

Cross multiplying, $9bh^3 = aK - 9ah^3 \Rightarrow h^3 = \dfrac{aK}{9(a+b)} = \dfrac{27\left(21^2 \cdot 28 \cdot 16\right)}{9(125)} = \dfrac{2^6 3^3 7^3}{5^3} \Rightarrow h = AE = \underline{\boldsymbol{\dfrac{84}{5}}}$.

B) Given: $AC = x + 3y + 1$, $BC = 2x + y - 3$, $r_{ic} = x - y$, and $d_{cc} = x + 5y$



Since the hypotenuse $\overline{AB}$ is the diameter of the <u>circumscribed</u> circle, we have $AB = x + 5y$.
*** The diameter of the <u>inscribed</u> circle equals $\underline{\underline{BC + AC - AB}} = 2(x - y)$
Thus, $(2x + y - 3) + (x + 3y + 1) - (x + 5y) = 2(x - y)$ or
$2x - y - 2 = 2x - 2y \Rightarrow y = 2$
Substituting into expressions for the sides,
$AC = x + 7$, $BC = 2x - 1$ and $AB = x + 10$
Applying the Pythagorean Theorem,
$(x+7)^2 + (2x-1)^2 = (x+10)^2 \Leftrightarrow 5x^2 + 10x + 50 = x^2 + 20x + 100 \Leftrightarrow 4x^2 - 10x - 50 = 0$
$\Rightarrow$ $2x^2 - 5x - 25 = 0 \Leftrightarrow (2x+5)(x-5) = 0$, so $x = 5$.
$\Rightarrow AC = 12$, $BC = 9$ and $AB = 15$, producing a perimeter of <u>**36**</u>.
Alternately, we could add the expressions for AC, BC and AB to get $4x + 9y - 2$, producing 20 + 18 – 2 = <u>**36**</u>.

Alternately, after finding $y = 2$, I conjecture a 3-4-5 right triangle or a multiple thereof, where $\overline{AC}$ is actually longer than $\overline{BC}$! $\dfrac{2x-1}{x+7} = \dfrac{3}{4} \Rightarrow 8x - 4 = 3x + 21 \Rightarrow x = 5$.
$(x, y) = (5, 2) \Rightarrow 9 - 12 - 15$ Bingo!

***** Challenge: Justify the double underlined result above.**
The proof is included at the end of this solution key.

Team Round - continued

C) The lengths of each candle after x minutes are $16\left(1-\frac{x}{240}\right), 12\left(1-\frac{x}{360}\right)$

Since the tall (inexpensive) candle burns faster and the initial lengths are in a ratio of 4 : 3, we know that after S minutes, the inexpensive candle will be half as tall as the expensive candle.

$$16\left(1-\frac{S}{240}\right)=\frac{1}{2}\cdot 12\left(1-\frac{S}{360}\right) \Leftrightarrow 16-\frac{S}{15}=6-\frac{S}{60} \Leftrightarrow \frac{S}{15}-\frac{S}{60}=10 \Leftrightarrow 3S=600 \Leftrightarrow S=200$$

After T minutes, $16\left(1-\frac{T}{240}\right)+12\left(1-\frac{T}{360}\right)=10 \Leftrightarrow 16-\frac{T}{15}+12-\frac{T}{30}=10$

$\Leftrightarrow 28\cdot 30-3T=300 \Leftrightarrow \frac{T}{10}=28-10 \Leftrightarrow T=180$. Thus, $T-S=\underline{\mathbf{-20}}$.

D) Brute force:

$k =$	1	2	3	4	5	<u>*6*</u>
feet travelled in the kth second	2640	1320	660	330	165	82.5
Total feet travelled	2640	3960	4620	4950	5115	5197.5
Feet to go	2640	1320	660	330	165	82.5

$\frac{5197.5}{6}=866.25 \Rightarrow \underline{866}$. Note at an average speed of 866 ft/sec over 6 seconds, the projectile has travelled 5196 feet and is only 84 feet from the target, within the 88 foot requirement, so rounding <u>down</u> produced the correct nearest integer. Thus, $(k,S)=\underline{(\mathbf{6,866})}$

Solution #2:

After k seconds, the projectile has travelled $5280\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^k}\right)=5280\left(\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^k\right)}{1-\frac{1}{2}}\right)$.

$=5280\left(\frac{2^k-1}{2^k}\right)$. The remaining distance to the target is $5280\left(1-\frac{2^k-1}{2^k}\right)=5280\left(\frac{1}{2^k}\right)$

$\Leftrightarrow 2^5\cdot 3\cdot 5\cdot 11\cdot 2^{-k} \Leftrightarrow 3\cdot 5\cdot 11\cdot 2^{5-k}<88 \Leftrightarrow 3\cdot 5\cdot 2^{2-k}<1 \Leftrightarrow 15<\frac{1}{2^{2-k}}=2^{k-2}=\frac{2^k}{4}$

$\Leftrightarrow 2^k>60 \Rightarrow k=6$ (ft/sec)

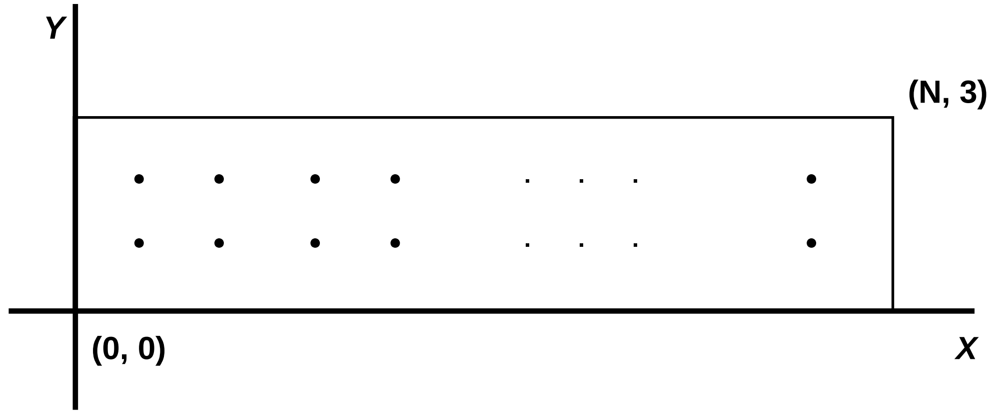
$$S=\frac{5280\left(\frac{2^k-1}{2^k}\right)}{k} \Rightarrow \frac{5280\left(\frac{63}{64}\right)}{6}=\frac{5197.5}{6}=866.25 \Rightarrow \underline{\mathbf{866}}$$

Team Round - continued

E) Let x denote the speed of the truck in feet/sec.
The length of each stripe is $x(1)$ feet.
The lengths of the gaps are $x(.3), x(.4), x(.5)$.
Thus, $AB = 4x + 1.2x = 104 \Rightarrow 5.2x = 104 \Rightarrow x =$ **<u>20</u>** feet/sec.
The stripes are 20 feet long and between x stripes there are $(x-1)$ intervals.
If the first interval is 3/10 sec., the second 4/10 sec., then the $(x-1)^{\text{st}}$ interval is $(x+1)$ sec.
We require that $20x + 20 \cdot \frac{1}{10}\left(3+4+...+(x+1)\right) \le 5280$
Applying the summation formula for the arithmetic series $\langle 1+2 \rangle + 3 + 4 + ... + (x+1)$, we have $20x + 2\left(\frac{(x+1)(x+2)}{2} - (1+2)\right) \le 5280$
$\Leftrightarrow 20x + \left(x^2 + 3x + 2 - 6\right) \le 5280 \Leftrightarrow x^2 + 23x = x(x+23) \le 5284$
By trial and error, let's start with $x = 60$ to get close to 5284 ($60 \cdot 83 = 4980$, $61 \cdot 84 = 5124$, $62 \cdot 85 = 5270$). Thus, **<u>62</u>** stripes and the intervening 61 intervals totals 5270 feet and the spraying mechanism is now in the off cycle for the next $\frac{64}{10} = 6.4$ seconds, resulting in a gap of $20(6.4) = 128$ feet and the 63rd stripe does not start until 5398 feet.

Team Round - continued

F)



There are two rows of lattice points, each of which contains $(N-1)$ points.

Consider points in the same row.

Each of the $(N-1)$ points must be connected to a point at least 2 units away.

The first point may be connected to the 3rd, 4th, 5th, etc. Moving only from left to right to avoid duplication, the second may be connected to the 4th, 5th 6th, etc.

The next to the next to last point may only be connected to the last point.

Thus, we have $(N-3)+(N-4)+...+1$ segments.

This is an arithmetic sequence with a sum of $\dfrac{(N-3)(N-2)}{2}$

$\Rightarrow$ a total of $(N-3)(N-2)$ for 2 rows.

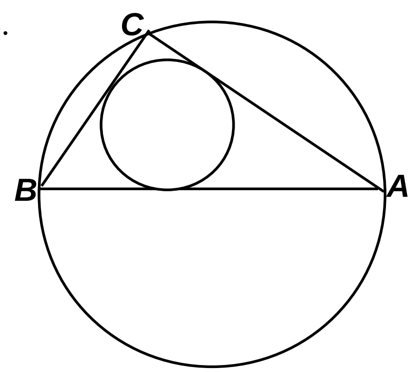
Consider points from different rows.

Any point in the bottom row can be connected to any point in the top row, excluding the point directly above it, producing $(N-1)(N-2)$ new segments.

Therefore, $(N-3)(N-2)+(N-1)(N-2)=(N-2)(2N-4)=450$

$\Rightarrow (N-2)^2=225 \Rightarrow N=15+2=\underline{\mathbf{17}}$

Team Round - continued



Here's the scoop on the challenge in the Team Round Question B.

We know that in any triangle $r_{ic} = \frac{A}{s}$. For right triangle ABC,

$$r_{ic} = \frac{\frac{1}{2}BC \cdot AC}{\frac{BC + AC + AB}{2}} \Rightarrow d_{ic} = \frac{2BC \cdot AC}{BC + AC + AB}$$

We wish to show that this is equivalent to $AC + BC - AB$

$\frac{2BC \cdot AC}{BC + AC + AB} = BC + AC - AB$ if and only if

$2BC \cdot AC = (BC + AC - AB)(BC + AC + AB)$ (since the denominator cannot be zero)

Multiplying out the right side, $(BC + AC)^2 - AB^2 = BC^2 + 2BC \cdot AC + AC^2 - AB^2$

Rearranging terms, we have

$(BC^2 + AC^2 - AB^2) + 2BC \cdot AC$

But, since ABC is a right triangle. $BC^2 + AC^2 = AB^2$ and the parenthesized expression is 0.
Therefore,

The diameter of a circle inscribed in any right triangle equals the sum of the lengths of the legs minus the length of the hypotenuse.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2015 ANSWERS

Round 1 Geometry Volumes and Surfaces

A) 5 B) $7\pi : 45$ C) 135π

Round 2 Pythagorean Relations

A) 442 B) 13 C) $3\sqrt{6}, 5\sqrt{6}$
(Both answers required.)

Round 3 Linear Equations

A) 4 B) 60 C) $(1250, 42)$

Round 4 Fraction & Mixed numbers

A) (8, 1), (–18, 3) B) –10 C) $(56,1)$

Round 5 Absolute value & Inequalities

A) 11 B) –1, –2 C) $-6 < x \le -\frac{8}{5}$ or $x > 5$

Round 6 Evaluations

A) 14 B) 10 C) 621

Team Round

A) $\frac{84}{5}$ or 16.8 D) (6, 866)

B) 36 E) 62

C) –20 F) 17